Interchange of Limit and Continuity

Thm 8.2.2 Let
- $f_n : A \to \mathbb{R}$ seq of continuous functions
- $f : A \to \mathbb{R}$
- $f_n \rightrightarrows f$ on A (converges uniformly)

Then f is continuous on A.

(i.e. uniform limit of continuous functions is continuous)

Pf: $f_n \rightrightarrows f$ on A

$\Leftrightarrow \quad \|f_n - f\|_A \to 0$

$\Rightarrow \quad \forall \varepsilon > 0, \exists H = H(\frac{\varepsilon}{3}) > 0$ s.t.

if $n \geq H$, $\|f_n - f\|_A < \frac{\varepsilon}{3}$

$$\|f_n - f\|_A = \sup\{|f_n(x) - f(x)| : x \in A\}$$

Now if $c \in A$, then $\forall x \in A$

$$\begin{aligned} |f(x) - f(c)| &\leq |f(x) - f_H(x)| + |f_H(x) - f_H(c)| + |f_H(c) - f(c)| \\ &\leq \|f_H - f\|_A + |f_H(x) - f_H(c)| + \|f_H - f\|_A \\ &< \frac{2\varepsilon}{3} + |f_H(x) - f_H(c)| \end{aligned}$$

Since f_H is continuous, $\exists \delta_\varepsilon(c) > 0$ such that

if $|x - c| < \delta_\varepsilon$, then $|f_H(x) - f_H(c)| < \varepsilon/3$.

Therefore, we have proved that

$\forall \varepsilon > 0,\ \exists \delta_\varepsilon(c) > 0$ s.t.

if $|x-c| < \delta_\varepsilon$,

$$|f(x)-f(c)| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$\therefore$ f is continuous at c.

Since $c \in A$ is arbitrary, f is continuous on A. #

(In this case, $\lim\limits_{x\to c} \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{x\to c} f(x) = f(c) = \lim\limits_{n\to\infty} f_n(c) = \lim\limits_{n\to\infty} \lim\limits_{x\to c} f_n(x)$)

Interchange of Limit and Derivative

<u>Thm 8.2.3</u> Let

- I be a bounded interval ($a<b$ finite numbers, $[a,b]$, $(a,b]$, $[a,b)$, (a,b))
- $f_n : I \to \mathbb{R}$ seq. of functions
- $\exists x_0 \in I$ such that $f_n(x_0)$ converges as $n \to +\infty$.
- f_n' exists on I (f_n' may not be continuous)
- $f_n' \rightrightarrows g$ on I for some function g (uniform convergent)

Then $\exists$ differentiable $f : I \to \mathbb{R}$

such that

- $f_n \rightrightarrows f$ on I, and
- $f' = g$ $\left(\left(\lim\limits_{n\to\infty} f_n\right)' = \lim\limits_{n\to\infty} f_n' \right)$

<u>Remark</u>: Since f_n' is not assumed to be continuous, f_n' may not integrable and hence the Fundamental Thm of Calculus may not applicable.

<u>Pf</u>: Let $m, n \in \mathbb{N}$, f_m' & f_n' exist

$\Rightarrow f_m - f_n$ is differentiable

Mean Value Thm $\Rightarrow$ if $x \in I$, then

$$(f_m - f_n)(x) - (f_m - f_n)(x_0) = (f_m' - f_n')(y)\,(x - x_0)$$

for some y between x & x_0,

where x_0 is the pt such that $(f_n(x_0))$ converges.

$$\therefore \quad f_m(x) - f_n(x) = f_m(x_0) - f_n(x_0) + (f_m'(y) - f_n'(y))(x - x_0)$$

$$\Rightarrow \quad |f_m(x) - f_n(x)| \le |f_m(x_0) - f_n(x_0)| + |f_m'(y) - f_n'(y)|\,|x - x_0|$$

$$\le |f_m(x_0) - f_n(x_0)| + \|f_m' - f_n'\|_I\,(b-a),$$

where $a < b$ are the endpts of I.

Taking sup over $x \in I$, we have

$$\|f_m - f_n\|_I \le |f_m(x_0) - f_n(x_0)| + \|f_m' - f_n'\|_I\,(b-a) \quad \text{——}(*),$$

(To be cont'd)