Ch 8 Sequences of Functions

§8.1 Pointwise and Uniform Convergence

Def: Let $A \subseteq \mathbb{R}$ be a set.

If $\forall n \in \mathbb{N} = \{1, 2, 3, \cdots\}$, there is a function

$$f_n : A \to \mathbb{R}$$

Then (f_n) is called a <u>sequence of functions on A</u> (to $\mathbb{R}$).

Remark: If (f_n) is a seq. of functions on A, then

$\forall x \in A$, $(f_n(x))$ is a sequence of numbers in $\mathbb{R}$.

Def 8.1.1 (<u>Pointwise Convergence</u>)

Let
- (f_n) be a sequence of functions on $A \subseteq \mathbb{R}$,
- $f : A_0 \to \mathbb{R}$, where $A_0 \subseteq A$

We say that the <u>sequence (f_n) converges on A_0 to f</u>

if $$\lim_{n\to\infty} f_n(x) = f(x), \quad \forall x \in A_0.$$

In this case,
- f is called the <u>limit on A_0 of the sequence (f_n)</u>.
- (f_n) is said to be <u>convergent on A_0</u>, or

  (f_n) <u>converges pointwise on A_0</u>.

<u>Remarks</u> (i) Usually, we choose

$$A_0 = \{ x \in A : (f_n(x)) \text{ converges} \}$$

(ii) Symbols:

$\left\{ \begin{array}{l} \bullet\ f = \lim f_n \quad \text{on } A_0, \text{ or} \\ \bullet\ f_n \to f \quad \text{on } A_0 \end{array} \right.$ ($f = \lim (f_n)$ in the Textbook)

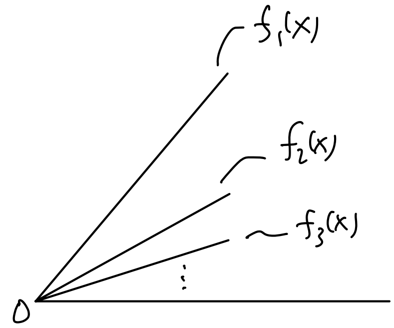
or

$\left\{ \begin{array}{l} \bullet\ f(x) = \lim f_n(x) \quad \text{for } x \in A_0, \text{ or} \\ \bullet\ f_n(x) \to f(x) \quad \text{for } x \in A_0 \end{array} \right.$

<u>Eg 8.1.2</u>

(a) $f_n(x) = \frac{x}{n}$, $\forall x \in \mathbb{R}$;

$f(x) = 0$, $\forall x \in \mathbb{R}$



Then $\lim_{n\to\infty} \frac{x}{n} = 0$, $\forall x \in \mathbb{R}$, implies

$\lim f_n = f \qquad (A_0 = \mathbb{R})$

(i.e. $\lim_{n\to\infty} \frac{x}{n} = 0$)

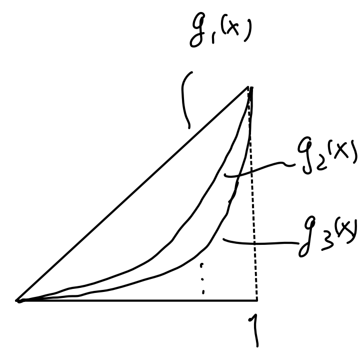
(b) $g_n(x) = x^n$, $\forall x \in \mathbb{R} \quad (n = 1, 2, 3, \cdots)$

If $|x| < 1$, then $x^n \to 0$ as $n \to \infty$

If $|x| > 1$, then x^n diverges.

and • $1^n = 1$, $\forall n = 1, 2, 3 \ldots$

• $(-1)^n$ diverges



$\left(\therefore A_0 = \{ x \in \mathbb{R} : -1 < x \leq 1 \} \right)$

and $\quad x^n \longrightarrow g(x) = \begin{cases} 0, & -1 < x < 1 \\ 1, & x = 1 \end{cases}$ on $(-1, 1]$

($\uparrow$ discontinuous at $x = 1$)

(c) Let $h_n(x) = \dfrac{x^2 + nx}{n}$, $\forall x \in \mathbb{R}$ and $\quad$ (see Textbook for graphs)

$h(x) = x$, $\forall x \in \mathbb{R}$

Then $\forall x \in \mathbb{R}$, $\lim_{n \to \infty} h_n(x) = \lim_{n \to \infty} \left(\frac{x^2}{n} + x \right) = x = h(x)$

$(\therefore A_0 = \mathbb{R})$

(d) $F_n(x) = \frac{1}{n} \sin(n(x+1))$, $\forall x \in \mathbb{R}$, and $\quad$ (see Textbook for graphs)

$F(x) = 0$, $\forall x \in \mathbb{R}$

Since $\forall x \in \mathbb{R}$

$$| F_n(x) - F(x) | = \frac{1}{n} | \sin(n(x+1)) | \leq \frac{1}{n} \to 0 \quad \text{as } n \to \infty$$

$\therefore F_n \to F$ on $\mathbb{R}$ $\quad$ (i.e. $A_0 = \mathbb{R}$)

<u>Lemma 8.1.3</u> A seq. $f_n : A \to \mathbb{R}$ <u>converges</u> to $f : A_0 \to \mathbb{R}$ $(A_0 \subseteq A)$ if and only if $\forall \varepsilon > 0$ and $\forall x \in A_0$, $\exists K(\varepsilon, x) \in \mathbb{N}$ s.t. $| f_n(x) - f(x) | < \varepsilon$, $\forall n \geq K(\varepsilon, x)$.

Pf: This follows from the definition of "limit of sequences".

Note that for different $x \in A_0$, $(f_n(x))$ are different sequences and hence the number K in the def of "limit of seq" depends on x too.

Therefore, we have a natural number $K(\varepsilon, x)$ (not just $K(\varepsilon)$ in general).

<u>eg 8.1.2(b)</u> For $|x|<1$, $|g_n(x) - g(x)| = |x^n| = |x|^n < \varepsilon$

Suppose $\varepsilon < 1$, then $n \log|x| < \log \varepsilon$

(note both $\log\varepsilon, \log|x| < 0$) $\Rightarrow n > \dfrac{\log 1/\varepsilon}{\log 1/|x|}$

$\therefore$ One need to choose $K(\varepsilon, x) = \left[\dfrac{\log 1/\varepsilon}{\log 1/|x|}\right] + 1$

(largest integer $\le \dfrac{\log 1/\varepsilon}{\log 1/|x|}$, i.e. integer part of the number)

which <u>depends on x</u>, and

$K(\varepsilon, x) \to +\infty$ as $|x| \to 1$

$\therefore$ Can't choose $K(\varepsilon)$ that works $\forall x \in (-1, 1]$

<u>eg 8.1.2(d)</u> $\forall x \in \mathbb{R}$, $|F_n(x) - F(x)| \le \frac{1}{n} < \varepsilon$ $\left(\Rightarrow n > \frac{1}{\varepsilon}\right)$

Only need to choose $K(\varepsilon) = \left[\frac{1}{\varepsilon}\right] + 1$

which is <u>independent of x</u> and works for all $x \in \mathbb{R}$.

Def 8.1.4 (Uniform Convergence)

A seq. $f_n : A \to \mathbb{R}$ converges uniformly on $A_0 \subseteq A$ to a function

$$f : A_0 \to \mathbb{R}$$

if $\forall \varepsilon > 0$, $\exists K(\varepsilon) \in \mathbb{N}$ (depends on ε, but not on $x \in A_0$)

s.t. if $n \geq K(\varepsilon)$, then

$$|f_n(x) - f(x)| < \varepsilon, \ \forall x \in A_0.$$

Remarks:

(i) In this case, we say that

(f_n) is uniformly convergent on A_0, and denoted by

- $f_n \rightrightarrows f$ on A_0 or
- $f_n(x) \rightrightarrows f(x)$ for $x \in A_0$

(Or in some other books, " $f_n \to f$ uniformly on A_0 ")

(ii) uniform convergence $\Longrightarrow$ pointwise convergence

i.e. " $f_n \rightrightarrows f$ on A_0 " $\Longrightarrow$ " $f_n \to f$ on A_0 "

(Easy from the definitions)

Lemma 8.1.5: $f_n : A \to \mathbb{R}$ does not converge uniformly on $A_0 \subseteq A$ to $f : A_0 \to \mathbb{R}$

$\Leftrightarrow$ $\exists$
- $\varepsilon_0 > 0$,
- a subsequence (f_{n_k}) of (f_n), and
- a seq. $x_k \in A_0$

such that $|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$, $\forall k = 1, 2, 3, \cdots$

Pf: Negation of Def. 8.1.4:

$\exists \varepsilon_0 > 0$, such that $\forall k \,(\leftrightarrow K(\varepsilon_0)) \in \mathbb{N}$,

the statement

"if $n \geq k$, then $|f_n(x) - f(x)| < \varepsilon_0$, $\forall x \in A_0$"

doesn't hold.

i.e. $\exists n_k (\geq k) \in \mathbb{N}$ s.t.

"$|f_{n_k}(x) - f(x)| < \varepsilon_0$, $\forall x \in A_0$" doesn't hold.

$\therefore \exists x_k \in A_0$ s.t. $|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$

All together, $\exists \varepsilon_0 > 0$, (f_{n_k}) subseq & $(x_k) \subset A_0$ s.t.

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0 . \quad ※$$

Eg 8.1.6

(a) eg 8.1.2 (a) $f_n(x) = \frac{x}{n}$, $f(x)=0$ $(A_0 = \mathbb{R})$

Consider $n_k = k$, $x_k = k \in \mathbb{R}$. Then

$$\left| f_{n_k}(x_k) - f(x_k) \right| = \left| \frac{x_k}{n_k} - 0 \right| = \left| \frac{k}{k} \right| = 1$$

$\therefore$ Choosing $\varepsilon_0 = 1$, then Lemma 8.1.5 $\Rightarrow$ $f_n \not\rightrightarrows f$ on $\mathbb{R}$

(b) eg 8.1.2 (b) $g_n(x) = x^n$, $g(x) = \begin{cases} 0, & |x|<1 \\ 1, & x=1 \end{cases}$, $A_0 = (-1, 1]$

Consider $n_k = k$, $x_k = \left(\frac{1}{2}\right)^{1/k}$ $(|x_k| < 1)$

Then $\left| g_{n_k}(x_k) - g(x_k) \right| = \left| \left[\left(\frac{1}{2}\right)^{1/k} \right]^k - 0 \right| = \frac{1}{2}$, $\forall k$

Choosing $\varepsilon_0 = 1/2$, Lemma 8.1.5 $\Rightarrow$ $x^n \not\rightrightarrows g$ on $(-1, 1]$.

(c) eg 8.1.2 (c) $h_n(x) = \frac{x^2 + nx}{n}$, $h(x) = x$, $A_0 = \mathbb{R}$

Consider $n_k = k$, $x_k = -k$,

Then
$$\left| h_{n_k}(x_k) - h(x_k) \right| = \left| \frac{(x_k)^2 + n_k x_k}{n_k} - x_k \right|$$
$$= \frac{(-k)^2 + k \cdot (-k)}{k} - (-k)$$
$$= k \geq 1 \qquad (\rightarrow \infty)$$

Choosing $\varepsilon_0 = 1$, lemma 8.1.5 $\Rightarrow$ $h_n \not\rightrightarrows h$ on $\mathbb{R}$.

<u>Def 8.1.7</u> (<u>Uniform Norm</u>) (<u>supnorm</u> in some other books)

If $\varphi : A \to \mathbb{R}$ is <u>bounded on A</u> (i.e. $\varphi(A)$ is a bounded subset of $\mathbb{R}$),

then we define the <u>uniform norm</u> of φ <u>on A</u> by

$$\|\varphi\|_A = \sup\{|\varphi(x)| : x \in A\}.$$

<u>Remark</u>: $\|\varphi\|_A \leq \varepsilon \iff |\varphi(x)| \leq \varepsilon, \ \forall x \in A.$

<u>Lemma 8.1.8</u>: $f_n \rightrightarrows f$ on $A \iff \|f_n - f\|_A \to 0.$

<u>Pf</u>: ($\Rightarrow$) $f_n \rightrightarrows f$ on A.

By Def 8.1.4, $\forall \varepsilon > 0$, $\exists K(\frac{\varepsilon}{2}) \in \mathbb{N}$

s.t. if $n \geq K(\frac{\varepsilon}{2})$, then

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}, \ \forall x \in A$$

$\therefore \forall \varepsilon > 0$, $\exists N(\varepsilon) = K(\frac{\varepsilon}{2}) \in \mathbb{N}$ s.t. if $n \geq N(\varepsilon)$

$$\|f_n - f\|_A \leq \frac{\varepsilon}{2} < \varepsilon \qquad \text{(by remark above)}$$

i.e. $\|f_n - f\|_A \to 0$ as $n \to \infty$.

($\Leftarrow$) If $\|f_n - f\|_A \to 0$. Then $\forall \varepsilon > 0$, $\exists K(\varepsilon) \in \mathbb{N}$ s.t.

if $n \geq K(\varepsilon)$, $\|f_n - f\|_A < \varepsilon$.

$\Rightarrow$ $|f_n(x) - f(x)| < \varepsilon, \ \forall x \in A.$

$\therefore f_n \rightrightarrows f$ on A. $\#$

Eg 8.1.9

(a) eg 8.1.2 (a), $f_n(x) = \frac{x}{n}$ on $\mathbb{R}$, $f(x) = 0$, on $\mathbb{R}$.

$f_n(x) - f(x) = \frac{x}{n}$ is unbounded, $\|f_n - f\|_{\mathbb{R}}$ is not defined.

However, if one consider only on the interval $A = [0,1]$.

Then $f_n(x) - f(x) = \frac{x}{n}$ is bounded on $[0,1]$,

and $\|f_n - f\|_{[0,1]} = \sup\{|\frac{x}{n}| : x \in [0,1]\}$

$$= \frac{1}{n} \quad (\to 0 \text{ as } n \to \infty)$$

$\therefore \ f_n|_{[0,1]} \rightrightarrows \underset{f}{\overset{\parallel}{0}}$ on $[0,1]$

(in fact $f_n \rightrightarrows f$ on any bounded subset, but $\not\rightrightarrows$ on unbounded subset)

(b) eg 8.1.2 (b), consider only on $[0,1] \subseteq A_0$. $\quad A_0 = (-1,1]$

Then $g_n(x) = x^n$, $g(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1. \end{cases}$

$$\|g_n - g\|_{[0,1]} = \sup\{|g_n(x) - g(x)| : x \in [0,1]\}$$

$$= \sup\left\{|x^n - g(x)| = \begin{cases} x^n, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}\right\}$$

$$= 1 \quad (\text{since } x^n \to 1 \text{ as } x \to 1^-)$$

$\|g_n - g\|_{[0,1]} \not\to 0$, $\therefore \ g_n \not\rightrightarrows g$ on $[0,1]$.

(c) eg 8.1.2 (c). $f_n(x) = \frac{x^2+nx}{n}$, $f(x) = x$ on $\mathbb{R}$

But $f_n(x) - f(x) = \frac{x^2}{n}$ is not bounded on $\mathbb{R}$.

$\therefore$ $\|f_n - f\|_{\mathbb{R}}$ doesn't define

But $f_n(x) - f(x) = \frac{x^2}{n}$ is bounded on $[0,8]$, and

$$\|f_n - f\|_{[0,8]} = \sup\left\{\left|\frac{x^2}{n}\right|, x \in [0,8]\right\} = \frac{64}{n}$$

$$\to 0 \text{ as } n \to \infty$$

$\therefore$ $f_n \rightrightarrows f$ on $[0,8]$ (but not on $\mathbb{R}$)

(d) eg 8.1.2(d) $F_n(x) = \frac{1}{n}\sin(n(x+1))$, $F(x) = 0$ on $\mathbb{R}$.

$|F_n(x) - F(x)| \leq \frac{1}{n}$, $\forall x \in \mathbb{R}$

$\Rightarrow$ $\|F_n - F\|_{\mathbb{R}} \leq \frac{1}{n}$ (in fact $\|F_n - F\| = \frac{1}{n}$ (Ex!))

$$\to 0 \text{ as } n \to \infty$$

$\therefore$ $F_n \rightrightarrows F$ on $\mathbb{R}$.

(e) $A = [0,1]$, $G_n(x) = x^n(1-x)$.

Clearly $G_n(x) \to 0$ $\forall x \in [0,1]$ (Ex!)

$\therefore$ G_n converges pointwisely to $G(x) \equiv 0$ on $A = [0,1]$.

To see whether G_n converges uniformly to G on $[0,1]$,

we calculate $\|G_n - G\|_{[0,1]}$:

$\forall x \in [0,1]$, $\quad |G_n(x) - G(x)| = x^n(1-x) \geq 0$

which is 0 at $x = 0, 1$.

$\therefore$ For interior max: $x \neq 0, 1$

$$0 = (x^n(1-x))' = nx^{n-1}(1-x) - x^n$$

$$= x^{n-1}(n - (n+1)x)$$

$G(x) \geq 0$ on $[0,1]$ & $G(0) = G(1) = 0$

$$\Rightarrow \quad x = \frac{n}{n+1} \quad \text{(only critical pt, hence "maximum")}$$

and
$$\|G_n - G\|_{[0,1]} = \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right)$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{n+1}$$

Note that $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$, we have

$$\|G_n - G\|_{[0,1]} \to 0 \quad \text{as } n \to \infty$$

$\therefore$ G_n converges uniformly to G on $[0,1]$. #

<u>Thm 8.1.10</u> (<u>Cauchy Criterion for Uniform Convergence</u>)

Let f_n be a seq. of <u>bounded</u> functions on A. Then f_n <u>converges uniformly</u> to a <u>bounded</u> function f on A

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists H(\varepsilon) \in \mathbb{N}$ s.t. $\forall m, n \geq H(\varepsilon)$,

$$\|f_m - f_n\|_A < \varepsilon.$$

Pf: ($\Rightarrow$) f_n converges uniformly to f on A (both f_n, f bdd)

$\Rightarrow \|f_n - f\|_A \to 0$ (Lemma 8.1.8)

$\therefore \forall \varepsilon > 0, \exists K(\varepsilon/2) \in \mathbb{N}$ s.t.

if $n \geq K(\varepsilon/2)$, then $\|f_n - f\|_A < \varepsilon/2$.

Hence letting $H(\varepsilon) = K(\varepsilon/2)$, we have

$\forall m, n \geq H(\varepsilon)$, $\|f_n - f\|_A < \varepsilon/2$ & $\|f_m - f\|_A < \varepsilon/2$

$$\Rightarrow \|f_m - f_n\|_A = \sup\{|f_m(x) - f_n(x)| : x \in A\}$$
$$\leq \sup\{|f_m(x) - f(x)| + |f_n(x) - f(x)| : x \in A\}$$
$$\leq \sup\{|f_m(x) - f(x)| : x \in A\} + \sup\{|f_n(x) - f(x)| : x \in A\}$$
$$= \|f_m - f\|_A + \|f_n - f\|_A < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

($\Leftarrow$) Conversely, if $\forall \varepsilon > 0, \exists H(\varepsilon) > 0$ s.t.

$\forall m, n \geq H(\varepsilon)$, $\|f_m - f_n\|_A < \varepsilon$.

Then $\forall x \in A$, $|f_m(x) - f_n(x)| \leq \|f_m - f_n\|_A < \varepsilon$ —— $(*)$

$\Rightarrow (f_n(x))$ is a Cauchy sequence.

By completeness of $\mathbb{R}$ (Thm 3.5.5), $f_n(x)$ is convergent.

Since the limit depends on x, we denote it by

$f(x) \stackrel{\text{def}}{=} \lim_{n \to \infty} f_n(x)$. ($f(x)$ is the pointwise limit of $f_n(x)$)

Then letting $m \to \infty$ in $(*)$, we have

$$|f(x) - f_n(x)| \leq \varepsilon, \ \forall\, x \in A.$$

i.e. $\forall \varepsilon > 0$, $\exists\, H(\varepsilon) \in \mathbb{N}$ s.t.

$$\text{if } n \geq H(\varepsilon), \ |f(x) - f_n(x)| \leq \varepsilon, \ \forall\, x \in A.$$

Since $\varepsilon > 0$ is arbitrary, this shows that f_n converges uniformly to f on A. ※

§8.2 Interchange of Limits

Eg 8.2.1

(a) (Eg 8.1.2(b)) $g_n(x) = x^n$ on $[0,1]$

$$g_n(x) \to g(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases} \quad \text{pointwise}$$

g ← discontinuous

$$\lim_{n\to\infty} \lim_{\substack{x\to 1 \\ (x\neq 1)}} g_n(x) = \lim_{n\to\infty} \lim_{\substack{x\to 1 \\ (x\neq 1)}} x^n = \lim_{n\to\infty} 1 = 1$$

$$\lim_{\substack{x\to 1 \\ (x\neq 1)}} \lim_{n\to\infty} g_n(x) = \lim_{\substack{x\to 1 \\ (x\neq 1)}} g(x) = 0 \quad (\text{since } g(x) = 0, \forall x < 1)$$

$$\therefore \lim_{n\to\infty} \lim_{\substack{x\to 1 \\ (x\neq 1)}} g_n(x) \neq \lim_{\substack{x\to 1 \\ (x\neq 1)}} \lim_{n\to\infty} g_n(x)$$

$\therefore$ Can't change limits for "pointwise convergence".

(b) (Same example) $g_n'(x) = n x^{n-1}$

$$g'(x) = \begin{cases} 0, & 0 \le x < 1 \\ \text{doesn't exist}, & x = 1 \end{cases}$$

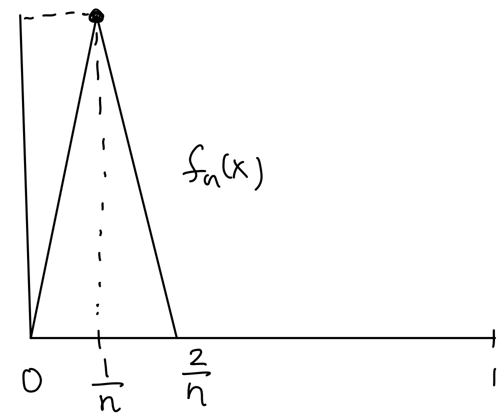
$\therefore$ "Pointwise limit" of sequence of differentiable functions may <u>not</u> be differentiable.

(c)

$$f_n(x) = \begin{cases} n^2 x & , \ 0 \le x \le \frac{1}{n} \\ -n^2\left(x - \frac{2}{n}\right) & , \ \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & , \ \frac{2}{n} \le x \le 1 \end{cases}$$

(well-defined at $x = \frac{1}{n}$ and $x = \frac{2}{n}$)

$(n \ge 2)$



It is easy to prove

$\lim_{n\to\infty} f_n(x) = 0$, $\forall x \in [0,1]$

$\therefore$ $f_n \to 0$ pointwisely

As f_n is cts, f_n is Riemann integrable and

$$\int_0^1 f_n = 1, \ \forall n \ge 2.$$

$$\therefore \quad 1 = \lim_{n\to\infty} \int_0^1 f_n \ne \int_0^1 \lim_{n\to\infty} f_n = 0.$$

$\therefore$ Integral of pointwise limit $\ne$ limit of integrals.

(d) Let $h_n(x) = 2nx e^{-nx^2}$, $x \in [0,1]$.

Then $\int_0^1 h_n = \int_0^1 2nxe^{-nx^2} dx = \int_0^1 (-e^{-nx^2})' dx$

$$= -e^{-nx^2}\Big|_0^1 = 1 - e^{-n}$$

$\therefore$ $\lim_{n\to\infty} \int_0^1 h_n = 1$

But $\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} 2nxe^{-nx^2} = 0$ $\quad \forall x \in [0,1]$ (Ex!)

$$\therefore \quad \int_0^1 \lim_{n\to\infty} h_n = 0 \ne \lim_{n\to\infty} \int_0^1 h_n.$$