Thm 7.4.8 (Integrability Criterion)

Let $f : [a,b] \to \mathbb{R}$ be bounded.

Then f is Darboux integrable

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists$ partition $\mathcal{P}_\varepsilon$ of $[a,b]$ such that

$$U(f; \mathcal{P}_\varepsilon) - L(f; \mathcal{P}_\varepsilon) < \varepsilon.$$

Pf: ($\Rightarrow$) f Darboux integrable

$$\Rightarrow L(f) = U(f).$$

Now $\forall \varepsilon > 0$, $\exists$ partition $\mathcal{P}_1$ of $[a,b]$ s.t.

$$L(f) - \frac{\varepsilon}{2} < L(f; \mathcal{P}_1) \quad \left(\text{as } L(f) = \sup\{L(f;\mathcal{P}) : \mathcal{P} \in \mathscr{P}([0,1])\}\right),$$

and partition $\mathcal{P}_2$ of $[a,b]$ s.t.

$$U(f; \mathcal{P}_2) < U(f) + \frac{\varepsilon}{2} \quad \left(\text{as } U(f) = \inf\{U(f;\mathcal{P}) : \mathcal{P} \in \mathscr{P}([0,1])\}\right)$$

Then the partition $\mathcal{P}_\varepsilon = \mathcal{P}_1 \cup \mathcal{P}_2$ is a refinement of $\mathcal{P}_1$ & $\mathcal{P}_2$, and hence by Lemmas 7.4.1 & 7.4.2

$$L(f) - \frac{\varepsilon}{2} < L(f; \mathcal{P}_1) \le L(f; \mathcal{P}_\varepsilon)$$

$$\le U(f; \mathcal{P}_\varepsilon) \le U(f; \mathcal{P}_2) < U(f) + \frac{\varepsilon}{2}$$

$$\Rightarrow U(f; \mathcal{P}_\varepsilon) - L(f; \mathcal{P}_\varepsilon) < U(f) + \frac{\varepsilon}{2} - \left(L(f) - \frac{\varepsilon}{2}\right)$$

$$= \varepsilon \quad (\text{as } U(f) = L(f))$$

($\Leftarrow$) For the converse, we observe $\forall$ partition $\mathcal{P}_\varepsilon$,

$$L(f;\mathcal{P}_\varepsilon) \le L(f) \quad \& \quad U(f) \le U(f;\mathcal{P}_\varepsilon)$$

$$\therefore 0 \le U(f) - L(f) \le U(f;\mathcal{P}_\varepsilon) - L(f;\mathcal{P}_\varepsilon) < \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, $U(f) = L(f)$

$\therefore$ f is Darboux integrable. #

Cor 7.4.9 Let $f : [a,b] \to \mathbb{R}$ bounded

If $\mathcal{P}_n$, $n = 1, 2, \cdots$, is a sequence of partitions of I s.t.

$$\lim_{n\to\infty} \left(U(f;\mathcal{P}_n) - L(f;\mathcal{P}_n) \right) = 0,$$

then f is (Darboux) integrable &

$$\int_a^b f = \lim_{n\to\infty} L(f;\mathcal{P}_n) = \lim_{n\to\infty} U(f;\mathcal{P}_n)$$

Pf: $\forall \varepsilon > 0$, $\exists n_\varepsilon > 0$ s.t.

$$0 \le U(f;\mathcal{P}_n) - L(f;\mathcal{P}_n) < \varepsilon, \quad \forall n \ge n_\varepsilon$$

Just pick one of the $\mathcal{P}_n$, $n \ge n_\varepsilon$ (says $\mathcal{P}_{n_\varepsilon}$) as $\mathcal{P}_\varepsilon$

and use the Integrability Criterion (Thm 7.4.8) #

Thm 7.4.10 Let $f:[a,b]\to\mathbb{R}$ be either <u>continuous</u> or <u>monotone</u>.

Then f is Darboux <u>integrable</u> on $[a,b]$.

<u>Pf</u>: Let $\mathcal{P}_n = (x_0, x_1, \cdots, x_n)$ be uniform partition of $[a,b]$ s.t.

$$x_k - x_{k-1} = \frac{b-a}{n}.$$

(1) If f is continuous, then

$$M_k = \sup\{f(x) : [x_{k-1}, x_k]\} = f(v_k) \quad \text{for some } v_k \in [x_{k-1}, x_k]$$

$$m_k = \inf\{f(x) : [x_{k-1}, x_k]\} = f(u_k) \quad \text{for some } u_k \in [x_{k-1}, x_k]$$

Then

$$L(f;\mathcal{P}_n) = \sum_k m_k (x_k - x_{k-1}) = \sum_k f(u_k)(x_k - x_{k-1})$$
$$= \int_a^b \alpha_\varepsilon$$

where α_ε is the step function (& n s.t. $\frac{b-a}{n} < \delta_\varepsilon$)

as in the proof of Thm 7.2.7.

and

$$U(f;\mathcal{P}_n) = \sum_k M_k (x_k - x_{k-1}) = \sum_k f(v_k)(x_k - x_{k-1})$$
$$= \int_a^b \omega_\varepsilon$$

where ω_ε is the step function (& n s.t. $\frac{b-a}{n} < \delta_\varepsilon$)

as in the proof of Thm 7.2.7.

$$\Rightarrow \quad U(f;\mathcal{P}_n) - L(f;\mathcal{P}_n) = \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$$

$\therefore$ Cor 7.4.9 $\Rightarrow$ f is Darboux integrable.

(2) If f is monotone (may assume increasing).

Then $M_k = \sup\{f(x) : [x_{k-1}, x_k]\} = f(x_k)$

$m_k = \inf\{f(x) : [x_{k-1}, x_k]\} = f(x_{k-1})$

and

$$L(f;\mathcal{P}_n) = \sum_k f(x_{k-1})(x_k - x_{k-1}) = \int_a^b \alpha$$

$$U(f;\mathcal{P}_n) = \sum_k f(x_k)(x_k - x_{k-1}) = \int_a^b \omega$$

where α, ω are functions as in the proof of Thm 7.2.8

$$\Rightarrow \quad U(f;\mathcal{P}_n) - L(f;\mathcal{P}_n) = \int_a^b (\omega - \alpha)$$

$$= \frac{b-a}{n}(f(b) - f(a))$$

$$\longrightarrow 0 \quad \text{as } n \to \infty$$

$\therefore$ Cor 7.4.9 $\Rightarrow$ f is Darboux integrable. ※

Thm 7.4.11 (Equivalence Theorem)

Let $f: [a,b] \to \mathbb{R}$ (bdd). Then

f is Darboux integrable $\Leftrightarrow$ f is Riemann integrable

In this case, the integrals equal.

Pf: ($\Rightarrow$) Assume f is Darboux integrable

By Thm 7.4.8 (Integrability Criterion),

$\forall \varepsilon > 0$, $\exists$ partition $\mathcal{P}_\varepsilon$ of $[a,b]$ s.t.

$$U(f;\mathcal{P}_\varepsilon) - L(f;\mathcal{P}_\varepsilon) < \varepsilon.$$

If $\mathcal{P}_\varepsilon = \{[x_{k-1}, x_k]\}_{k=1}^n$, define step functions α_ε & ω_ε

s.t. $\alpha_\varepsilon(x) = m_k = \inf_{[x_{k-1},x_k]} f$, $\forall x \in [x_{k-1}, x_k)$ if $k=1,\dots,n-1$, $[x_{n-1}, x_n]$ if $k=n$

and

$\omega_\varepsilon(x) = M_k = \sup_{[x_{k-1},x_k]} f$, $\forall x \in [x_{k-1}, x_k)$ if $k=1,\dots,n-1$, $[x_{n-1}, x_n]$ if $k=n$

Then $\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x)$ $\forall x \in [a,b]$.

and $$\int_a^b \alpha_\varepsilon = \sum_k m_k(x_k - x_{k-1}) = L(f;\mathcal{P}_\varepsilon)$$

$$\int_a^b \omega_\varepsilon = \sum_k M_k(x_k - x_{k-1}) = U(f;\mathcal{P}_\varepsilon)$$

$\Rightarrow$ $$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) = U(f;\mathcal{P}_\varepsilon) - L(f;\mathcal{P}_\varepsilon) < \varepsilon$$

$\therefore$ Squeeze Thm 7.2.1 $\Rightarrow$ $f \in \mathcal{R}[a,b]$.

($\Leftarrow$) If $f \in \mathcal{R}[a,b]$ with $A = \int_a^b f$

Then f is bounded on $[a,b]$ and

$\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ s.t. if $\dot{\mathcal{P}}$ satisfies $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$,

then $\quad |S(f; \dot{\mathcal{P}}) - A| < \varepsilon$.

Let $\mathcal{P} = (x_0, x_1, \cdots, x_n)$ be a partition with $\|\mathcal{P}\| < \delta_\varepsilon$.

By definition of $M_k = \sup_{[x_{k-1}, x_k]} f$, $\exists t_k \in [x_{k-1}, x_k]$

such that $\quad f(t_k) > M_k - \frac{\varepsilon}{b-a}$.

Similarly, $\exists t_k' \in [x_{k-1}, x_k]$ s.t.

$$f(t_k') < m_k + \frac{\varepsilon}{b-a}, \quad \text{where } m_k = \inf_{[x_{k-1}, x_k]} f$$

Then the tagged partition $\dot{\mathcal{P}} = \{[x_{k-1}, x_k], t_k\}_{k=1}^n$ has

Riemann sum
$$\begin{aligned} S(f; \dot{\mathcal{P}}) &= \sum_{k=1}^n f(t_k)(x_k - x_{k-1}) \\ &> \sum_{k=1}^n \left(M_k - \frac{\varepsilon}{b-a}\right)(x_k - x_{k-1}) \\ &= \sum_{k=1}^n M_k(x_k - x_{k-1}) - \frac{\varepsilon}{b-a} \sum_{k=1}^n (x_k - x_{k-1}) \\ &= U(f; \mathcal{P}) - \varepsilon \end{aligned}$$

Using $|S(f; \dot{\mathcal{P}}) - A| < \varepsilon$, we have

$$U(f;\mathcal{P}) < S(f;\dot{\mathcal{P}}) + \varepsilon < A + 2\varepsilon.$$

Hence $\quad U(f) < A + 2\varepsilon.$

Since $\varepsilon > 0$ is arbitrary, $\quad U(f) \le A.$

Similarly for the tagged partition $\dot{\mathcal{P}}' = \{[x_{k-1}, x_k], t_k'\}_{k=1}^{n}$,

$$S(f;\dot{\mathcal{P}}') = \sum_{k=1}^{n} f(t_k')(x_k - x_{k-1})$$

$$< \sum_{k=1}^{n}\left(m_k + \frac{\varepsilon}{b-a}\right)(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} m_k(x_k - x_{k-1}) + \frac{\varepsilon}{b-a}\sum_{k=1}^{n}(x_k - x_{k-1})$$

$$= L(f;\mathcal{P}) + \varepsilon$$

$$\Rightarrow \quad L(f;\mathcal{P}) > S(f;\dot{\mathcal{P}}') - \varepsilon > A - 2\varepsilon.$$

$$\Rightarrow \quad L(f) > A - 2\varepsilon, \quad \forall \varepsilon > 0$$

$$\Rightarrow \quad L(f) \ge A.$$

Therefore $\quad A \le L(f) \le U(f) \le A$

$\Rightarrow$ f is Darboux integrable,

and the Darboux integral $= A$. #

§7.5 Approximate Integration (Omitted)