Thm 7.3.14 (Composition Theorem)

Let $\begin{cases} \bullet\ f \in \mathcal{R}[a,b] \text{ with } f([a,b]) \subset [c,d], \\ \bullet\ \varphi : [c,d] \to \mathbb{R} \text{ continuous} \end{cases}$

$$\left(\begin{tikzcd} {[a,b]} \arrow[r, "f"] \arrow[rr, bend right, "\varphi\circ f"'] & {[c,d]} \arrow[r, "\varphi"] & \mathbb{R} \end{tikzcd} \right)$$

Then $\varphi \circ f \in \mathcal{R}[a,b]$.

("φ cts" is needed, see ex. 7.3.22, Homework 6)

Pf: Let D = set of discontinuity of f on $[a,b]$,

D_1 = set of discontinuity of $\varphi \circ f$ on $[a,b]$.

If $u \in [a,b] \setminus D$, then f is continuous at u,
Since φ is cts, $\varphi \circ f$ is also continuous at u.

$\therefore \quad u \in [a,b] \setminus D_1$

Therefore $[a,b] \setminus D \subset [a,b] \setminus D_1$,
and hence $D_1 \subset D$.

Note that $f \in \mathcal{R}[a,b]$. Lebesgue's Integrable Criterion

$\Rightarrow$ D is of measure zero.

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists$ countable collection of open intervals $\{I_k\}_{k=1}^{n}$

s.t. $D \subset \bigcup_{k=1}^{\infty} I_k$ & $\sum_{k=1}^{\infty} \text{length}(I_k) \le \varepsilon$.

Since $D_1 \subset D$, we have $D_1 \subset \bigcup_{k=1}^{\infty} I_k$ & $\sum_{k=1}^{\infty} \text{length}(I_k) \le \varepsilon$

$\therefore$ D_1 is also of measure zero.

Using Lebesgue's Integrability criterion again, we have

$$\varphi \circ f \in \mathcal{R}[a,b].$$

$\#$

(In this proof, we showed that a <u>subset</u> of a <u>null</u> set is also a <u>null</u> set.)

<u>Cor 7.3.15</u> If $f \in \mathcal{R}[a,b]$, then $|f| \in \mathcal{R}[a,b]$

and $$\left| \int_a^b f \right| \le \int_a^b |f| \le M(b-a)$$

for any $M \ge 0$ s.t. $|f(x)| \le M$ on $[a,b]$

<u>Pf</u>: $f \in \mathcal{R}[a,b] \Rightarrow f$ is bounded

$\Rightarrow \quad |f(x)| \le M$ on $[a,b]$ for some $M > 0$.

Then $f([a,b]) \subset [-M, M]$ and

$$|\cdot| : [-M, M] \to \mathbb{R} \text{ is continuous.}$$

By Thm 7.3.14, $|f| \in \mathcal{R}[a,b]$

Since $-|f|(x) \le f(x) \le |f|(x)$, $\forall x \in [a,b]$,

Thm 7.1.5(c) $\Rightarrow \quad -\int_a^b |f| \le \int_a^b f \le \int_a^b |f|$

$$\therefore \quad \left| \int_a^b f \right| \le \int_a^b |f|.$$

Similarly, $|f|(x) \le M \quad \forall x \in [a,b]$

$$\Rightarrow \quad \int_a^b |f| \le \int_a^b M = M(b-a)$$

$\#$

Thm 7.3.16 (The Product Thm) If f & $g \in \mathcal{R}[a,b]$, then $fg \in \mathcal{R}[a,b]$.

Pf: $f \in \mathcal{R}[a,b] \Rightarrow \exists M>0$ s.t. $f([a,b]) \subset [-M,M]$.

and $\varphi(t) = t^2 : [-M,M] \to \mathbb{R}$ is cts

$\therefore$ $f^2 \in \mathcal{R}[a,b]$.

Similarly $g \in \mathcal{R}[a,b] \Rightarrow g^2 \in \mathcal{R}[a,b]$.

By Thm 7.1.5(b), $f, g \in \mathcal{R}[a,b] \Rightarrow f+g \in \mathcal{R}[a,b]$.

Hence $(f+g)^2 \in \mathcal{R}[a,b]$.

Therefore, Thm 7.1.5 again, $fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right] \in \mathcal{R}[a,b]$ $\#$

Thm 7.3.17 (Integration by Parts)

Let
- F, G be differentiable on $[a,b]$
- $f = F'$, $g = G' \in \mathcal{R}[a,b]$

Then fG, $Fg \in \mathcal{R}[a,b]$ and

$$\int_a^b fG = FG\Big|_a^b - \int_a^b Fg$$

Pf: F, G diff on $[a,b] \Rightarrow F, G$ cts on $[a,b]$

$\Rightarrow F, G \in \mathcal{R}[a,b]$ (Thm 7.2.7)

Product Thm 7.3.16 then implies fG & $Fg \in \mathcal{R}[a,b]$.

And product rule Thm 6.1.3 (c),

$$(FG)' = F'G + FG' = fG + Fg \in \mathcal{R}[a,b]$$

Fundamental Thm 7.3.1 $\Rightarrow$

$$\int_a^b (FG)' = FG\Big|_a^b$$

$$\therefore \quad \int_a^b fG + \int_a^b Fg = FG\Big|_a^b$$

※

Thm 7.3.18 (Taylor's Thm with Remainder (Integral Form))

Suppose
- $f : [a,b] \to \mathbb{R}$
- $f', \cdots, f^{(n)}, f^{(n+1)}$ exist on $[a,b]$
- $f^{(n+1)} \in \mathcal{R}[a,b]$

Then $f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \cdots + \frac{f^{(n)}(a)}{n!}(b-a) + R_n$

where $$R_n = \frac{1}{n!}\int_a^b f^{(n+1)}(t)(b-t)^n\, dt.$$

Pf: $$R_n = \frac{1}{n!}\int_a^b f^{(n+1)}(t)(b-t)^n\, dt \quad \text{(is defined by Product Thm)}$$

$$= \int_a^b \left(f^{(n)}\right)'(t)\left(\frac{(b-t)^n}{n!}\right) dt \quad \text{(Integration by Parts) Thm 7.3.17}$$

$$= f^{(n)}(t)\frac{(b-t)^n}{n!}\Big|_a^b - \int_a^b f^{(n)}(t)\left[-\frac{(b-t)^{n-1}}{(n-1)!}\right] dt$$

$$= -\frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{1}{(n-1)!}\int_a^b f^{(n)}(t)(b-t)^{n-1}\,dt$$

$$= -\frac{f^{(n)}(a)}{n!}(b-a)^n + R_{n-1}$$

$$= -\frac{f^{(n)}(a)}{n!}(b-a)^n - \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + R_{n-2} \quad \left(\begin{array}{c}\text{Same}\\ \text{calculation}\end{array}\right)$$

$$\vdots$$

$$= -\left(\frac{f^{(n)}(a)}{n!}(b-a)^n + \cdots + \frac{f'(a)}{1!}(b-a)\right) + R_0$$

where $R_0 = \frac{1}{0!}\int_a^b f'(t)(b-t)^0\,dt = \int_a^b f' = f(b) - f(a)$

So we are done. $\#$

§7.4 The Darboux Integral

Def (Upper and Lower Sums)

Let
- $f : [a,b] \to \mathbb{R}$ bounded
- $\mathcal{P} = (x_0, x_1, \cdots, x_n)$ partition of $[a,b]$
- $m_k = \inf\{ f(x) : x \in [x_{k-1}, x_k] \}$ (exist because of "bddness")

  $M_k = \sup\{ f(x) : x \in [x_{k-1}, x_k] \}$

The
- lower sum of f corresponding to $\mathcal{P}$ is defined to be

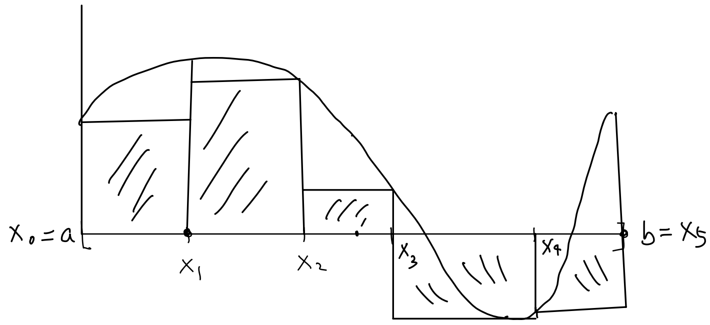
$$L(f;\mathcal{P}) = \sum_{k=1}^{n} m_k (x_k - x_{k-1}) \;;$$

- upper sum of f corresponding to $\mathcal{P}$ is defined to be

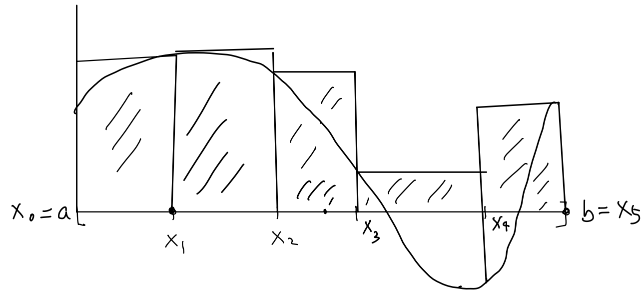
$$U(f;\mathcal{P}) = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$$

Remarks (i) upper and lower sums are not Riemann sums in general, (because m_k, M_k may not attained at any point in $[x_{k-1}, x_k]$) unless the function f is cts.

(ii) On one hand, $L(f;\mathcal{P})$ and $U(f;\mathcal{P})$ are simpler because they do not involve the infinitely many possibilities of tags. But on the other hand, inf and sup are harder to handle than values of a function.



lower sum $L(f;\mathcal{P})$



upper sum $U(f;\mathcal{P})$

Lemma 7.4.1 If $f:[a,b]\to\mathbb{R}$ is bounded and $\mathcal{P}$ is a partition of $[a,b]$.

Then $L(f;\mathcal{P}) \leq U(f;\mathcal{P})$

Pf: (Easy) $\quad m_k = \inf_{[x_{k-1},x_k]} f \leq \sup_{[x_{k-1},x_k]} f = M_k$

$$\Rightarrow \quad L(f;\mathcal{P}) = \sum_k m_k (x_k - x_{k-1}) \leq \sum_k M_k (x_k - x_{k-1}) = U(f;\mathcal{P})$$

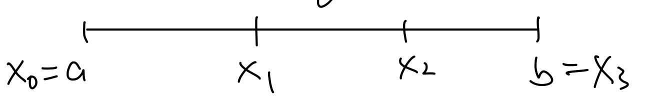
\#

Def: If $\mathcal{P}$, Q are partitions of $[a,b]$ and $\mathcal{P}\subset Q$, then we say that Q is a refinement of $\mathcal{P}$.

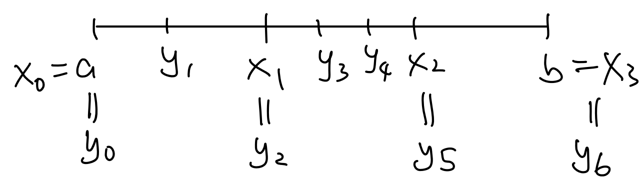
Remark: If $\mathcal{P} = (x_0, x_1, \cdots, x_n)$ and $Q = (y_0, y_1, \cdots, y_m)$, then Q is a refinement of $\mathcal{P}$ if $x_k \in \mathcal{P}$, $\forall k = 0,1,\cdots,n \Rightarrow x_k \in Q$

(i.e. $x_k = y_\ell$ for some $\ell = 0,1,\cdots,m$)

$\mathcal{P}$



Q



In other wads, subinterval $[x_{k-1}, x_k]$ of $\mathcal{P}$ is further subdivided in Q: $\quad [x_{k-1}, x_k] = [y_{j-1}, y_j] \cup \cdots \cup [y_{h-1}, y_h]$.

Lemma 7.4.2 If
- $f:[a,b] \to \mathbb{R}$ is bounded
- $\mathcal{P}$ is a partition of $[a,b]$
- Q is a refinement of $\mathcal{P}$.

Then $L(f;\mathcal{P}) \leq L(f;Q)$ and $U(f;Q) \leq U(f,\mathcal{P})$

Pf: Special case Q is a refinement of $\mathcal{P}$ by adjoining one point.

Let $\mathcal{P} = (x_0, x_1, \cdots, x_n)$ and

$$Q = (x_0, x_1, \cdots, x_{k-1}, z, x_k, \cdots x_n)$$

Then
$$m_k' = \inf\{f(x) : x \in [x_{k-1}, z]\}$$
$$\geq \inf\{f(x) : x \in [x_{k-1}, x_k]\} = m_k$$

&
$$m_k'' = \inf\{f(x) : x \in [z, x_k]\}$$
$$\geq \inf\{f(x) : x \in [x_{k-1}, x_k]\} = m_k$$

$$\Rightarrow \quad L(f;\mathcal{P}) = \sum_{i \neq k} m_i(x_i - x_{i-1}) + m_k(x_k - x_{k-1})$$
$$= \sum_{i \neq k} m_i(x_i - x_{i-1}) + m_k(z - x_{k-1}) + m_k(x_k - z)$$
$$\leq \sum_{i \neq k} m_i(x_i - x_{i-1}) + m_k'(z - x_{k-1}) + m_k''(x_k - z)$$
$$= L(f;Q)$$

Similarly $U(f;\mathcal{P}) \geq U(f;Q)$ (ex!)

General Case

If Q is a refinement of $\mathcal{P}$, then Q can be obtained from $\mathcal{P}$ by adjoining a finite number of points to $\mathcal{P}$ one at a time.

Hence, repeating the special case (or using induction), we have $L(f;\mathcal{P}) \leq L(f;Q)$

and $U(f;Q) \leq U(f;\mathcal{P})$ #

Lemma 7.4.3 Let • $f:[a,b] \to \mathbb{R}$ be bounded.

Then $L(f;\mathcal{P}_1) \leq U(f;\mathcal{P}_2)$

for any partitions $\mathcal{P}_1, \mathcal{P}_2$ of $[a,b]$.

Pf: Let $Q = \mathcal{P}_1 \cup \mathcal{P}_2$.

Then Q is a refinement of $\mathcal{P}_1$ and also of $\mathcal{P}_2$.

Hence Lemma 7.4.1 & Lemma 7.4.2

$$\Rightarrow \quad L(f;\mathcal{P}_1) \leq L(f;Q) \leq U(f;Q) \leq U(f;\mathcal{P}_2)$$

#

<u>Notation</u>: Let $\mathcal{P}([a,b])$ = set of partitions of $[a,b]$.

<u>Def 7.4.4</u> Let $f:[a,b]\to\mathbb{R}$ be <u>bounded</u>.

The <u>lower integral of f on I</u> is the number

$$L(f) = \sup\{L(f;\mathcal{P}) : \mathcal{P}\in\mathcal{P}[a,b]\}$$

and the <u>upper integral of f on I</u> is the number

$$U(f) = \inf\{U(f;\mathcal{P}) : \mathcal{P}\in\mathcal{P}[a,b]\}$$

<u>Thm 7.4.5</u> Let $f:[a,b]\to\mathbb{R}$ be bounded. Then $L(f)$ and $U(f)$ of f on $[a,b]$ exist and $L(f)\le U(f)$

<u>Pf</u>: • <u>$L(f)$ and $U(f)$ exist</u>

f bounded $\Rightarrow$ $m_I = \inf\{f(x) : x\in I=[a,b]\}$ &

$M_I = \sup\{f(x) : x\in I=[a,b]\}$ exist

It is clear that $\forall\, \mathcal{P}\in\mathcal{P}([a,b])$

$$m_I(b-a)\le L(f;\mathcal{P})\le U(f;\mathcal{P})\le M_I(b-a)$$

$\therefore$ $L(f)$ and $U(f)$ exist

(and satisfy $m_I(b-a)\le L(f)$ & $U(f)\le M_I(b-a)$)

- $\underline{L(f) \le U(f)}$

By Lemma 7.4.3,

$$L(f;\mathcal{P}_1) \le U(f;\mathcal{P}_2) \quad \text{for any partitions } \mathcal{P}_1 \text{ \& } \mathcal{P}_2$$

Fixing $\mathcal{P}_2$ and letting $\mathcal{P}_1$ runs through $\mathscr{P}([a,b])$, we have

$$L(f) = \sup\{L(f;\mathcal{P}_1) : \mathcal{P}_1 \in \mathscr{P}([a,b])\} \le U(f;\mathcal{P}_2).$$

Then letting $\mathcal{P}_2$ runs through $\mathscr{P}([a,b])$, we have

$$L(f) \le \inf\{U(f;\mathcal{P}_2) : \mathcal{P}_2 \in \mathscr{P}([a,b])\} = U(f)$$

\#

<u>Def 7.4.6</u> Let $f : [a,b] \to \mathbb{R}$ be <u>bounded</u>. Then f is said to be <u>Darboux integrable</u> on $[a,b]$ if $\underline{L(f) = U(f)}$.

In this case, the <u>Darboux integral</u> of f over $[a,b]$ is defined to be the value $L(f) = U(f)$.

<u>Remark</u>: We'll use the same notation $\int_a^b f$ or $\int_a^b f(x)\,dx$ for Darboux integral (since it is equal to the Riemann integral (Thm 7.4.11))

Eg 7.4.7

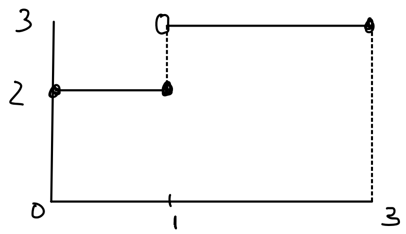
(a) A constant function is Darboux integrable

In fact, if $f(x) \equiv c$ on $[a,b]$ & $\mathcal{P}$ is any partition of $[a,b]$,

then $$L(f;\mathcal{P}) = c(b-a) = U(f;\mathcal{P}) \quad (\text{Ex } 7.4.2)$$

$$\therefore \quad L(f) = c(b-a) = U(f) \quad ※$$

(b) $g:[0,3]\to\mathbb{R}$ defined by $g(x)=\begin{cases} 3, & 1<x\leq 3 \\ 2, & 0\leq x\leq 1\end{cases}$ (eg 7.1.4(b))



(is (Riemann) integrable & $\int_0^3 g = 8$.)

Using Darboux's approach, we only need to prove

$$L(f) = U(f)$$

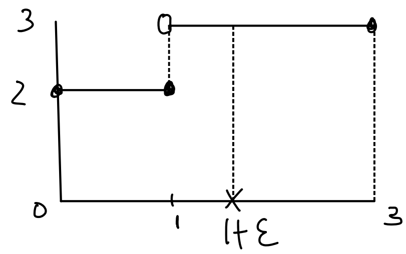
No need to check whether they exist (since we have Thm 7.4.5)

As $L(f) = \sup\{\text{of something}\}$ &

$U(f) = \inf\{\text{of something}\}$

we only need to find sequence/family of partitions that can prove the required result, no need to consider all partitions.

g is clearly bounded.



$\forall \varepsilon > 0$, consider the partition

$$\mathcal{P}_\varepsilon = (0, 1, 1+\varepsilon, 3)$$

Then

$$\big(\swarrow \sup\{g(x) : x \in [1, 1+\varepsilon]\} = 3 \big)$$

$$U(g; \mathcal{P}_\varepsilon) = 2 \cdot (1-0) + 3 \cdot (1+\varepsilon - 1) + 3 \cdot (3 - (1+\varepsilon))$$

$$= 2 + 3\varepsilon + 6 - 3\varepsilon = 8$$

$$\Rightarrow \quad U(g) \le 8 \qquad \big(U(g) = \inf\{ U(g; \mathcal{P}) : \mathcal{P} \in \mathcal{P}([0,3])\} \big)$$

And

$$L(g; \mathcal{P}_\varepsilon) = 2 \cdot (1-0) + 2 \cdot (1+\varepsilon - 1) + 3 \cdot (3 - (1+\varepsilon))$$

$$\big(\nwarrow \inf\{g(x) : x \in [1, 1+\varepsilon]\} = 2 \big)$$

$$= 2 + 2\varepsilon + 6 - 3\varepsilon = 8 - \varepsilon$$

$$\Rightarrow \quad 8 - \varepsilon \le L(g) \qquad \big(L(g) = \sup\{ L(g; \mathcal{P}) : \mathcal{P} \in \mathcal{P}([0,3])\} \big)$$

Hence, Thm 7.4.5 $\Rightarrow$

$$8 - \varepsilon \le L(g) \le U(g) \le 8$$

Since $\varepsilon > 0$ is arbitrary, we have

$$L(g) = U(g) = 8$$

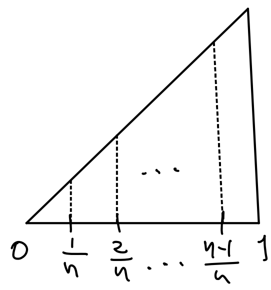
$\therefore$ g is Darboux integrable & $\int_a^b g = 8$

(Easier than "Riemann")

(c) $h(x) = x$ on $[0,1]$ is integrable

h is clearly bounded.

Let $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\}$.



Then $U(h; P_n) = \frac{1}{n}\cdot\left(\frac{1}{n}-0\right) + \frac{2}{n}\cdot\left(\frac{2}{n}-\frac{1}{n}\right) + \cdots + 1\cdot\left(1-\frac{n-1}{n}\right)$

$$= \frac{1}{n^2}(1+2+\cdots+n) = \frac{n(n+1)}{2n^2} = \frac{1}{2}\left(1+\frac{1}{n}\right)$$

and $L(h; P_n) = 0\cdot\left(\frac{1}{n}-0\right) + \frac{1}{n}\cdot\left(\frac{2}{n}-\frac{1}{n}\right) + \cdots + \frac{n-1}{n}\cdot\left(1-\frac{n-1}{n}\right)$

$$= \frac{1}{n^2}(1+2+\cdots+(n-1)) = \frac{n(n-1)}{2n^2} = \frac{1}{2}\left(1-\frac{1}{n}\right)$$

$\therefore \quad \frac{1}{2}\left(1-\frac{1}{n}\right) \le L(h) \le U(h) \le \frac{1}{2}\left(1+\frac{1}{n}\right)$

Letting $n \to \infty$, we have $\quad L(h) = U(h) = \frac{1}{2}$

$\therefore \quad h(x) = x$ is Darboux integrable on $[0,1]$

& $\int_a^b h = \frac{1}{2}$.

(d) (Eg 7.2.2 (b), not integrable)

Dirichlet function $f(x) = \begin{cases} 1, & \text{if } x \text{ rational}, x \in [0,1] \\ 0, & \text{if } x \text{ irrational}, x \in [0,1]. \end{cases}$

[To prove non-integrable, we need to consider all partitions, as a seqence/family of partitions can only provide upper bound for $U(f)$ & lower bound for $L(f)$; not good enough to see $U(f) > L(f)$.]

f is clearly bounded: $0 \le f \le 1$.

Let $\mathcal{P} = (x_0, x_1, \cdots, x_n)$ be a partition of $[0,1]$.

Then for each subinterval $[x_{k-1}, x_k]$,

$\exists$ rational $r_k \in [x_{k-1}, x_k]$ and

irrational $t_k \in [x_{k-1}, x_k]$

$\Rightarrow$ $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} = f(r_k) = 1$ &

$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} = f(t_k) = 0$

$\therefore$ $U(f;\mathcal{P}) = \sum_k M_k(x_k - x_{k-1}) = \sum_k (x_k - x_{k-1}) = 1$, $\forall \mathcal{P}$

$\Rightarrow$ $U(f) = \inf\{U(f;\mathcal{P}) : \mathcal{P} \in \mathscr{P}([0,1])\} = 1$

And $L(f;\mathcal{P}) = \sum_k m_k(x_k - x_{k-1}) = 0$, $\forall \mathcal{P}$

$\Rightarrow$ $L(f) = \sup\{L(f;\mathcal{P}) : \mathcal{P} \in \mathscr{P}([0,1])\} = 0$.

$\therefore$ $U(f) = 1 > 0 = L(f)$

f is not Darboux integrable. #