Eg 7.3.11 $\mathbb{Q}_1$ = set of rational numbers in $[0,1]$ is a null set.
(set of measure zero)

Pf: $\mathbb{Q}_1$ is countable and can be written as

$$\mathbb{Q}_1 = \{r_1, r_2, r_3, \cdots\}$$

Given $\varepsilon > 0$, define open intervals

$$J_k = \left(r_k - \frac{\varepsilon}{2^{k+1}},\ r_k + \frac{\varepsilon}{2^{k+1}}\right), \quad k = 1, 2, \cdots$$

Clearly $r_k \in J_k$ and length of $J_k = \frac{\varepsilon}{2^k}$.

$\therefore$ $\mathbb{Q}_1 \subset \bigcup_{k=1}^{\infty} J_k$ and $\sum_{k=1}^{\infty}$ length of $J_k = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, $\mathbb{Q}_1$ is a null set.

Note: From the proof, it is clear that it doesn't use the fact that r_k are rational. Hence, the proof can be used to prove that:

$$\boxed{\text{Every countable set is a null set (set of measure zero)}}$$

("countable infinite" can be proved similarly, "countable finite" are included by dropping the tail of the infinite series)

Thm 7.3.12 (Lebesgue's Integrability Criterion)

A bounded function $f: [a,b] \to \mathbb{R}$ is Riemann integrable if and only if it is continuous almost everywhere on $[a,b]$

(Pf: Omitted. See App. C of the Textbook)

Eg 7.3.13

(a) Every step function on $[a,b]$ is bdd & has a finite set of points of discontinuity which is a set of measure zero and hence every step function on $[a,b]$ is Riemann integrable.

(b) Every monotone function on $[a,b]$ is Riemann integrable

In fact, monotone functions are bounded &

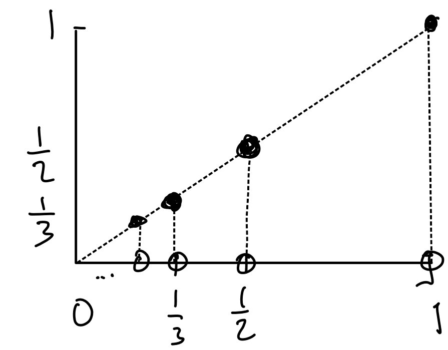
Thm 5.6.4 $\Rightarrow$ set of points of discontinuity of a monotonic function is countable.

Hence, it is a null set.

$\therefore$ Lebesgue's Integrability criterion $\Rightarrow$ it is Riemann integrable

(c) (eg 7.1.4(d))

$$G(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{1}{n} \ (n = 1, 2, \cdots) \\ 0, & \text{elsewhere on } [0,1] \end{cases}$$



is bounded, and

Set of discontinuity $= \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$

is countable hence measure zero.

Lebesque's Integrability criterion $\Rightarrow$ $G(x)$ is Riemann integrable

(d) (Eg 7.2.2 (b), not integrable)

Dirichlet function $f(x) = \begin{cases} 1, & \text{if } x \text{ rational}, x \in [0,1] \\ 0, & \text{if } x \text{ irrational}, x \in [0,1]. \end{cases}$

is bounded.

set of discontinuity $= [0,1]$ (discontinuous at every $x \in [0,1]$)

which can be shown that it is <u>not a null set</u> (Omitted)

$\therefore$ Lebesque's Integrability criterion $\Rightarrow$

Dirichlet function is not Riemann integrable.

(e) (eg 7.1.7) Thomae's function

$$h(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \in [0,1] \ \& \ \underset{\neq 0}{m}, n \text{ have no common factors } \in \mathbb{N} = \{1,2,3,\cdots\} \ (\gcd(m,n)=1) \\ 1, & \text{if } x = 0 \\ 0, & \text{if } x \text{ is irrational } \& \ x \in [0,1]. \end{cases}$$

is bounded & (by eg 5.1.6 (h))

set of discontinuity $= \mathbb{Q}_1$ (set of rational numbers in $[0,1]$)

which is of measure zero (eg 7.3.11)

$\therefore$ Lebesgue's Integrability criterion $\Rightarrow$

Thomae's function is Riemann integrable on $[0,1]$.