Cor 7.2.10 If $f \in \mathcal{R}[a,b]$ & $[c,d] \subset [a,b]$, then $f \in \mathcal{R}[c,d]$.

Pf : By Additivity Thm 7.2.8

$$f \in \mathcal{R}[a,b] \Rightarrow f \in \mathcal{R}[c,b] \Rightarrow f \in \mathcal{R}[c,d] \quad \#$$

Cor 7.2.11 If $f \in \mathcal{R}[a,b]$ & $a = c_0 < c_1 < \cdots < c_m = b$,

then $f|_{[c_{i-1}, c_i]} \in \mathcal{R}[c_{i-1}, c_i]$ and

$$\int_a^b f = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} f$$

(Pf : By Induction)

Def : If $f \in \mathcal{R}[a,b]$ and $\alpha, \beta \in [a,b]$ with $\alpha < \beta$,

we define $\int_\beta^\alpha f \overset{def}{=\!=} -\int_\alpha^\beta f$ and

$$\int_\alpha^\alpha f \overset{def}{=\!=} 0$$

Thm 7.2.13 If $f \in \mathcal{R}[a,b]$ and $\alpha, \beta, \gamma \in [a,b]$,

then $$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f \qquad \text{———} \qquad (*)$$

in the sense that the existence of any two of these integrals implies the third integral exists & $(*)$ holds

Pf: Clearly Thm 7.2.9 & Cor 7.2.11 implies the statement that

"the <u>existence</u> of any two of these integrals

$\Rightarrow$ the third integral exists".

Now if any two of α, β, γ equal,

then $(*)$ is trivially holds (check)

If α, β, γ are distinct, we consider

$$L(\alpha,\beta,\gamma) \overset{\text{def}}{=} \int_\alpha^\beta f + \int_\beta^\gamma f + \int_\gamma^\alpha f$$

$$= \int_\alpha^\beta f - \int_\gamma^\beta f - \int_\alpha^\gamma f$$

Clearly $L(\alpha,\beta,\gamma) = L(\beta,\gamma,\alpha) = L(\gamma,\alpha,\beta)$ (check!)

$$= -L(\alpha,\gamma,\beta) = -L(\gamma,\beta,\alpha) = -L(\beta,\alpha,\gamma)$$

(eg: $L(\alpha,\beta,\gamma) = \int_\alpha^\beta f + \int_\beta^\gamma f + \int_\gamma^\alpha f$

$$= -\int_\beta^\alpha f - \int_\gamma^\beta f - \int_\alpha^\gamma f = -L(\alpha,\gamma,\beta)\ )$$

By Additivity Thm 7.2.9, if $\alpha < \gamma < \beta$,

then $L(\alpha,\beta,\gamma) = \int_\alpha^\beta f - \left(\int_\alpha^\gamma f + \int_\gamma^\beta f\right) = 0$.

By the above, we have $L(\alpha,\beta,\gamma) = 0$

for all other situations: $\gamma<\beta<\alpha$, $\beta<\alpha<\gamma$

$\gamma<\alpha<\beta$, $\alpha<\beta<\gamma$, & $\beta<\gamma<\alpha$.

Hence $\forall \alpha, \beta, \gamma$,

$$0 = L(\alpha, \beta, \gamma) = \int_\alpha^\beta f - \left(\int_\alpha^\gamma f + \int_\gamma^\beta f \right)$$

i.e. $$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f$$ ※

§7.3 The Fundamental Theorem

Recall: A function $F:[a,b]\to\mathbb{R}$ is called an antiderivative

or a primitive of $f:[a,b]\to\mathbb{R}$ on $[a,b]$ if

$$F'(x)=f(x)\,,\quad \forall\, x\in[a,b]$$

(One sided derivatives at $x=a$ & $x=b$)

Thm 7.3.1 (Fundamental Theorem of Calculus (1^{st} Form))

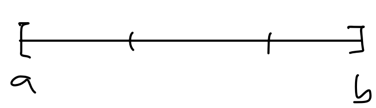
Suppose
- $f, F:[a,b]\to\mathbb{R}$ functions,
- E = finite set of $[a,b]$ (E the exceptional set)

such that
- (a) F is continuous on $[a,b]$,
- (b) $F'(x)=f(x)$ $\forall\, x\in[a,b]\setminus E$,
- (c) $f\in\mathcal{R}[a,b]$

Then

$$\boxed{\int_a^b f = F(b)-F(a)}$$

Pf: With the finite # of points in E,



$[a,b]$ is subdivided into finite

number of subintervals such that $F'(x)=f(x)$ on the

subintervals except possibly at endpoints.

Then by Thm 7.1.3 & Thm 7.2.9, one can reduce the proof of the Thm to the case that

$E = \{a, b\}$ two end points only

i.e. $F'(x) = f(x), \forall x \in (a,b)$.

needs assumption (a)

(Exercise 7.3.1 of the Textbook, using F cts & $\sum_{i=1}^{n} F(x_i) - F(x_{i-1}) = F(b) - F(a)$)

For this special case, consider any $\varepsilon > 0$.

Then $f \in \mathcal{R}[a,b]$ (assumption (c)) $\Rightarrow$

$\exists \, \delta_\varepsilon > 0$ such that

if $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ satisfies $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$, (any tags t_i)

then $\left| S(f, \dot{\mathcal{P}}) - \int_a^b f \right| < \varepsilon$. ——— $(*)$

By Mean Value Thm 6.2.4, $\exists \, u_i \in (x_{i-1}, x_i)$ s.t.

$$F(x_i) - F(x_{i-1}) = F'(u_i)(x_i - x_{i-1})$$

$$= f(u_i)(x_i - x_{i-1}), \quad \forall i = 1, \cdots, n$$

since $F' = f$ exists on (a,b) (assumption (b) of the special case)
(& cts on $[a,b]$)

Hence $F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]$

$$= \sum_{i=1}^{n} f(u_i)(x_i - x_{i-1})$$

Define the tagged partition $\dot{\mathcal{P}}_u = \{[x_{i-1}, x_i], u_i\}_{i=1}^{n}$

(same partition with new tags).

Then $\|\dot{\mathcal{P}}_u\| < \delta_\varepsilon$ and

$$F(b) - F(a) = S(f, \dot{\mathcal{P}}_u)$$

$$\therefore \quad \left| F(b) - F(a) - \int_a^b f \right| < \varepsilon, \quad \text{by } (*)$$

Since $\varepsilon > 0$ is arbitrary, $\int_a^b f = F(b) - F(a)$. ✱

<u>Remarks</u>: (i) If $E = \emptyset$, then assumption (b) $\Rightarrow$ assumption (a).

(ii) One may allow f defined on $[a,b]$ except finite number of points as one can extend f to all $x \in [a,b]$ by setting $f(c) = 0$ for $c \notin \text{domain}(f)$ originally.

(iii) F differentiable on $[a,b] \not\Rightarrow F' \in \mathcal{R}[a,b]$

$\therefore$ assumption (c) is not automatically satisfied even $E = \emptyset$ & assumption (b) is satisfied. (Eg 7.3.2 (e))

<u>Eg 7.3.2</u> (a)

- $F(x) = \frac{1}{2}x^2$, $\forall x \in [a,b]$ is continuous on $[a,b]$,
- $F'(x) = x$, $\forall x \in [a,b]$ ($\therefore E = \emptyset$)
- $F'(x) = x \in \mathcal{R}[a,b]$ (says by Thm 7.2.7, cts $\Rightarrow$ integrable)

$$\therefore \int_a^b x\,dx = F(b) - F(a) = \frac{1}{2}(b^2 - a^2).$$

(b) Suppose $[a,b]$ is a closed interval s.t. $\quad (\operatorname{Arctan} x = \tan^{-1} x)$

$G(x) = \operatorname{Arctan} x$ is defined on $[a,b]$ $\quad$ (for instance $[a,b] \subset (-\frac{\pi}{2}, \frac{\pi}{2})$)

Then $G'(x) = \dfrac{1}{x^2+1}$, $\forall x \in [a,b]$ & is <u>continuous</u> on $[a,b]$

$\therefore$ (b) satisfied with $E = \phi$. (with $f(x) = \frac{1}{x^2+1}$)

Hence (a) satisfied automatically. (G' exists $\Rightarrow$ G cts)

And Thm 7.2.7 $\Rightarrow$ (c) is also satisfied.

$$\therefore \quad \int_a^b \frac{dx}{x^2+1} = \operatorname{Arctan} b - \operatorname{Arctan} a.$$

(c) $A(x) = |x|$ for $x \in [-10, 10]$, cts.

(one can do any $[\alpha, \beta]$ with $\alpha, \beta > 0$)

Then

$$A'(x) = \begin{cases} 1, & \text{for } x \in (0, 10] \\ \text{doesn't exist}, & \text{for } x = 0 \\ -1, & \text{for } x \in [-10, 0) \end{cases}$$

Recall the signum function

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

$\therefore \quad A'(x) = \operatorname{sgn}(x) \quad \forall x \in [-10, 10] \setminus \{0\}$ (i.e. $E = \{0\}$)

Note that $sgn(x)$ equals a step function except at one point,

Thm 7.2.5 (& Thm 7.1.3) $\Rightarrow$ $sgn(x) \in \mathcal{R}[-10, 10]$.

Hence $\int_{-10}^{10} sgn(x)\,dx = A(10) - A(-10) = 10 - 10 = 0$.

(d) $H(x) = 2\sqrt{x}$ on $[0, b]$.

Then $H(x)$ cts on $[0, b]$,

$$H'(x) = \frac{1}{\sqrt{x}} \quad \forall\ x \in (0, b] \quad (E = \{0\})$$

Note that $h(x) = \frac{1}{\sqrt{x}}$ is unbounded on $[0, b]$,

$h \notin \mathcal{R}[0, b]$ (No matter how we define $H'(0)$)

$\therefore$ Fundamental Thm 7.3.1 doesn't apply!

(Need to consider improper integrals, which is equivalent to applying Thm 7.3.1 to $[\varepsilon, b]$, and then letting $\varepsilon \to 0$.)

(e)

$$K(x) = \begin{cases} x^2 \cos\left(\frac{1}{x^2}\right), & x \in (0, 1] \\ 0, & x = 0 \end{cases}$$

Then

$$K'(x) = \begin{cases} 2x\cos\frac{1}{x^2} + \frac{2}{x}\sin\left(\frac{1}{x^2}\right), & x \in (0, 1] \\ 0, & \text{if } x = 0 \quad (\text{eg } 6.1.7(c)) \end{cases}$$

That is, K differentiable on $[0, 1]$, & hence cts on $[0, 1]$.

However K' is unbounded and

therefore $K' \notin \mathcal{R}[0,1]$, assumption (c) doesn't satisfy!

<u>Def 7.3.3</u>: If $f \in \mathcal{R}[a,b]$, then the function defined by

$$F(z) = \int_a^z f \quad \text{for } z \in [a,b],$$

is called the <u>indefinite integral</u> of f with <u>basepoint</u> a.

(One may use other point as base point & is still called indefinite integral (Ex 7.3.6))

<u>Thm 7.3.4</u> If $f \in \mathcal{R}[a,b]$, then

$$F(z) = \int_a^z f \quad \text{is } \underline{\text{continuous}} \text{ on } [a,b]$$

and in fact, if $|f(x)| \leq M$, $\forall x \in [a,b]$, then

$$(*) \qquad |F(z) - F(w)| \leq M|z-w|, \quad \forall z, w \in [a,b].$$

<u>Remarks:</u> (i) M exists because $f \in \mathcal{R}[a,b] \Rightarrow f$ is bdd

(ii) $(*)$ is called a <u>Lipschitz condition</u>, much stronger than just continuity.

<u>Pf</u> $\forall z, w \in [a,b]$ with $w \leq z$, Additivity Thm 7.2.9 $\Rightarrow$

$$F(z) = \int_a^z f = \int_a^w f + \int_w^z f = F(w) + \int_w^z f$$

$$\therefore \quad F(z) - F(w) = \int_w^z f.$$

If $\qquad -M \leq f(x) \leq M, \quad \forall x \in [a,b],$

Thm 7.1.5 (c) $\Rightarrow \quad -M(z-w) \leq \int_w^z f \leq M(z-w)$

$$\therefore \quad |F(z) - F(w)| = \left| \int_w^z f \right| \leq M(z-w) = M|z-w|$$

(since $w \leq z$)

Clearly, the case $z \leq w$ follows immediately too. #

<u>Thm 7.3.5</u> (<u>Fundamental Theorem of Calculus (2nd Form)</u>)

Let $f \in \mathcal{R}[a,b]$ and <u>continuous at c</u>.

Then $F(z) = \int_a^z f$ is <u>differentiable at $z = c$</u> and

$$F'(c) = f(c).$$

<u>Pf</u> We'll prove only for the <u>right-hand derivative</u>

$$\lim_{h \to 0^+} \frac{F(c+h) - F(c)}{h} = f(c)$$

The left-hand derivative can be handled similarly.

Therefore, we assume $c \in [a,b)$.

Since f is continuous at c, $\forall \varepsilon > 0$, $\exists \eta_\varepsilon > 0$ s.t. if

(*) $|f(x) - f(c)| < \varepsilon$, $\forall x \in [c, c+\eta_\varepsilon)$. (consider only right side)

Let $h \in (0, \eta_\varepsilon)$, then Additivity Thm 7.2.9 (Cor 7.2.10)

$\Rightarrow f \in \mathcal{R}[a, c+h]$, $\mathcal{R}[a,c]$ & $\mathcal{R}[c, c+h]$ and

$$\int_a^{c+h} f = \int_a^c f + \int_c^{c+h} f$$

i.e. $$F(c+h) - F(c) = \int_c^{c+h} f$$

By (*) $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$, $\forall x \in [c, c+\eta_\varepsilon)$

we have $$(f(c) - \varepsilon) h \le \int_c^{c+h} f \le (f(c) + \varepsilon) h,$$

which implies $$f(c) - \varepsilon \le \frac{F(c+h) - F(c)}{h} \le f(c) + \varepsilon$$

$$\Rightarrow \left| \frac{F(c+h) - F(c)}{h} - f(c) \right| < \varepsilon, \quad \forall h \in (0, \eta_\varepsilon)$$

It proves that $$\lim_{h \to 0^+} \frac{F(c+h) - F(c)}{h} = f(c)$$

⋕

$\boxed{\begin{array}{l} \underline{\text{Thm 7.3.6}} \quad \text{If } f \text{ is } \underline{\text{continuous}} \text{ on } [a,b]\text{, then} \\ \quad \bullet\ F(x) = \int_a^x f \text{ is } \underline{\text{differentiable}} \text{ on } [a,b]\text{, and} \\ \quad \bullet\ F'(x) = f(x),\ \forall x \in [a,b] \end{array}}$

Pf: f cts on $[a,b] \Rightarrow f \in \mathcal{R}[a,b]$ & cts at every pt. $c \in [a,b]$ #

<u>Eg 7.3.7</u>

(a) $f(x) = \operatorname{sgn} x$ on $[-1,1]$.

Then
- $f \in \mathcal{R}[-1,1]$ (equals a step function except at a point)
- f <u>not</u> continuous at $x=0$, but continuous $\forall x \in [-1,1] \setminus \{0\}$.

Simply calculation: indefinite integral with basepoint -1 is

$$F(x) = \int_{-1}^{x} \operatorname{sgn}(x)\,dx = |x| - 1 \quad (\text{Ex!})$$

One can see that $F'(0)$ doesn't exist ("f cts at c" is a necessary condition)

and F is <u>not</u> an antiderivative of $f(x) = \operatorname{sgn}(x)$.

(b) Let h = Thomae's function

$\mathbb{N} = \{1,2,3,\cdots\}$

$$h(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \in [0,1] \text{ \& } m, n \text{ have no common factors } (\gcd(m,n)=1),\ m \neq 0 \\ 1, & \text{if } x = 0 \\ 0, & \text{if } x \text{ is irrational \& } x \in [0,1]. \end{cases}$$

Then by Eg 7.1.7, one concludes that

$$H(x) = \int_0^x h \equiv 0 \quad , \forall x \in [0,1]$$

$\Rightarrow$ $H'(x) = 0$ exists $\forall x \in [0,1]$

However, $H'(x) \neq h(x)$, $\forall$ rational $x \in [0,1]$.

Thm 7.3.8 (Substitution Theorem)

Let
- $f : I \to \mathbb{R}$ cts, (I = interval)
- $\varphi : [\alpha, \beta] \to \mathbb{R}$ st. $\varphi'(t)$ exists & cts $\forall t \in [\alpha, \beta]$.
  (i.e. φ has a continuous derivative)
- $\varphi([\alpha, \beta]) \subset I$

$$\left(\begin{tikzcd}[\alpha,\beta] \arrow[r, "\varphi"] \arrow[rr, bend right, "f\circ\varphi"'] & I \arrow[r, "f"] & \mathbb{R}\end{tikzcd} \right)$$

Then $\displaystyle \int_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$

Notes: (i) t & x in the formula are dummy variables, just using them for convenient in practice: thinking of change of variables $x = \varphi(t)$ (but it is not necessary a "change of variables" as φ is not assumed to be one-to-one and onto.)

In fact, the formula can be written as

$$\int_\alpha^\beta (f\circ\varphi)\cdot\varphi' = \int_{\varphi(\alpha)}^{\varphi(\beta)} f$$

(ii) The formula holds also for $\varphi(\beta) \leq \varphi(\alpha)$ as we defined before.

Pf of Thm 7.3.8 : Ex 7.3.17 (Easy application of Fundamental Thm & Chain rule)

Eg 7.3.9 Too easy, Omitted

Lebesgue's Integrability Criterion

Def 7.3.10

(a) A set $Z \subset \mathbb{R}$ is said to be a null set (set of measure zero) if $\forall \varepsilon > 0$, $\exists$ a countable collection $\{(a_k, b_k)\}_{k=1}^{\infty}$ of open intervals (could be overlapped) such that

$$Z \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k) \quad \text{and} \quad \sum_{k=1}^{\infty} (b_k - a_k) \leq \varepsilon$$

$\leftarrow$ length of interval (a_k, b_k)

(b) If $Q(x)$ is a statement about $x \in I$, we say that

"$Q(x)$ holds almost everywhere on I"

(or "$Q(x)$ holds for almost every (almost all) $x \in I$")

if $\exists$ a null set $Z \subset I$ s.t.

$$Q(x) \text{ holds } \forall\, x \in I \setminus Z.$$

In this case, we write $Q(x)$ for a.e. $x \in I$.

Remarks : (i) "null set" may means "empty set" for some people. So "set of measure zero" is used more often.

(ii) Def (a) means Z can be covered by a set of <u>arbitrary small</u> total length. (Kind of "length of $Z = 0$", but it is difficult to define "length" of arbitrary sets in $\mathbb{R}$.)