Thm 7.2.8 If $f:[a,b]\to\mathbb{R}$ is <u>monotone</u> on $[a,b]$, $(-\infty<a<b<+\infty)$

then $f\in\mathcal{R}[a,b]$.

Pf: Suppose f is <u>increasing</u> (<u>decreasing</u> are similar)

Take uniform partition $\mathcal{P}=\{[x_{i-1},x_i]\}_{i=1}^{n}$ such that

$$x_i-x_{i-1}=\frac{b-a}{n}\quad \forall i=1,2,\cdots,n \quad (\text{with } x_0=a)$$

Then $f(x_{i-1})\le f(x)\le f(x_i)$, $\forall x\in[x_{i-1},x_i]$ $(\forall i=1,\cdots,n)$

Define step functions

$$\alpha_n(x)=\begin{cases} f(x_{i-1}), & x\in[x_{i-1},x_i) \\ f(x_{n-1}), & x\in[x_{n-1},x_n] \end{cases}$$

and
$$\omega_n(x)=\begin{cases} f(x_i), & x\in[x_{i-1},x_i) \\ f(x_n), & x\in[x_{n-1},x_n] \end{cases}$$

Then
$$\alpha_n(x)\le f(x)\le\omega_n(x),\quad \forall x\in[a,b]$$

and
$$\int_a^b\alpha_n=\sum_{i=1}^{n}f(x_{i-1})(x_i-x_{i-1})$$
$$=\frac{b-a}{n}\left[f(x_0)+f(x_1)+\cdots+f(x_{n-1})\right]$$
$$\int_a^b\omega_n=\sum_{i=1}^{n}f(x_i)(x_i-x_{i-1})$$
$$=\frac{b-a}{n}\left[f(x_1)+f(x_2)+\cdots+f(x_n)\right]$$

$\therefore \int_a^b (\omega_n - \alpha_n) = \frac{b-a}{n}\left[f(x_n) - f(x_0)\right] = \frac{(b-a)(f(b)-f(a))}{n}$

Hence $\forall \varepsilon > 0$, $\exists\, n_\varepsilon > \frac{(b-a)(f(b)-f(a))}{\varepsilon}$ s.t.

$$\alpha_{n_\varepsilon}(x) \le f(x) \le \omega_{n_\varepsilon}(x), \ \forall x \in [a,b] \quad \&$$

$$\int_a^b (\omega_{n_\varepsilon} - \alpha_{n_\varepsilon}) < \varepsilon$$

$\therefore f \in \mathcal{R}[a,b]$ by Squeeze Thm 7.2.3 ※

<u>Thm 7.2.9</u> (<u>Additivity Theorem</u>)

Let $f : [a,b] \to \mathbb{R}$ & $c \in (a,b)$. $(a<b)$

Then $f \in \mathcal{R}[a,b] \iff f|_{[a,c]} \in \mathcal{R}[a,c]$ & $f|_{[c,b]} \in \mathcal{R}[c,b]$.

In this case $\int_a^b f = \int_a^c f + \int_c^b f$

<u>Pf</u> ($\Rightarrow$) By Cauchy Criterion (Thm 7.2.1)

$f \in \mathcal{R}[a,b]$

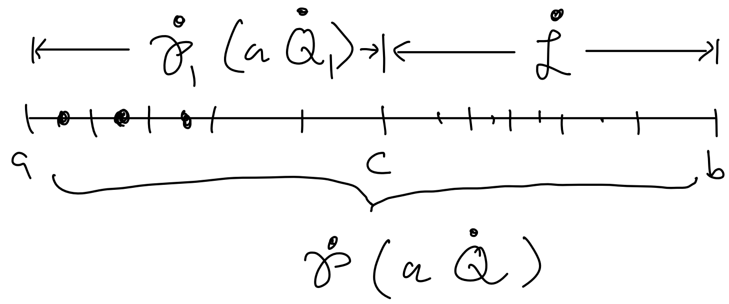
$\iff \forall \varepsilon > 0$, $\exists\, \eta_\varepsilon > 0$ s.t. $\forall \dot{\mathcal{P}}, \dot{Q}$ with $\|\dot{\mathcal{P}}\| < \eta_\varepsilon$ & $\|\dot{Q}\| < \eta_\varepsilon$

we have $|S(f, \dot{\mathcal{P}}) - S(f, \dot{Q})| < \varepsilon$. ——— $(*)_1$

Now we want to show that the same $\eta_\varepsilon > 0$ works for the restriction $f_1 = f|_{[a,c]} : [a,c] \to \mathbb{R}$.

Suppose $\dot{\mathcal{P}}_1$ & $\dot{Q}_1$ be two tagged partitions of $[a,c]$ with $\|\dot{\mathcal{P}}_1\| < \eta_\varepsilon$ & $\|\dot{Q}_1\| < \eta_\varepsilon$.

Define now tagged partitions $\dot{\mathcal{P}}$ & $\dot{Q}$ of $[a,b]$ by adding a tagged partition $\dot{\mathcal{L}}$ of $[c,b]$ with $\|\dot{\mathcal{L}}\| < \eta_\varepsilon$ to $\dot{\mathcal{P}}_1$ & $\dot{Q}_1$



Then clearly $\quad \|\dot{\mathcal{P}}\| < \eta_\varepsilon$ & $\|\dot{Q}\| < \eta_\varepsilon$

By $(*)_1$, $\quad |S(f,\dot{\mathcal{P}}) - S(f,\dot{Q})| < \varepsilon$.

On the other hand

$$S(f,\dot{\mathcal{P}}) = \underbrace{\sum_{x_i \le c} f(t_i)(x_i - x_{i-1})}_{\dot{\mathcal{P}}_1} + \underbrace{\sum_{x_{i-1} \ge c} f(t_i)(x_i - x_{i-1})}_{\dot{\mathcal{L}}}$$

and

$$S(f,\dot{Q}) = \underbrace{\sum_{x_i' \le c} f(t_i')(x_i' - x_{i-1}')}_{\dot{Q}_1} + \underbrace{\sum_{x_{i-1} \ge c} f(t_i)(x_i - x_{i-1})}_{\dot{\mathcal{L}}}$$

$\therefore \quad S(f, \dot{\mathcal{P}}) - S(f, \dot{Q}) = S(f_1, \dot{\mathcal{P}}_1) - S(f_1, \dot{Q}_1)$

$$\Rightarrow \qquad |S(f_1, \dot{\mathcal{P}}_1) - S(f_1, \dot{Q}_1)| < \varepsilon$$

Hence $f_1 : [a, c] \to \mathbb{R}$ satisfies Cauchy Criterion.

Therefore $f_1 \in \mathcal{R}[a, c]$.

Similarly, we have $\quad f_2 = f|_{[c,b]} \in \mathcal{R}[c, b]$.

($\Leftarrow$) Suppose $f_1 = f|_{[a,c]} \in \mathcal{R}[a,c]$ & $f_2 = f|_{[c,b]} \in \mathcal{R}[c,b]$.

Then Boundedness Thm 7.1.6 $\Rightarrow$ $f|_{[a,c]}$ & $f|_{[c,b]}$ are bdd.

$\Rightarrow$ f is bounded on $[a,b]$.

i.e. $\exists M > 0$ such that $|f(x)| \leq M$, $\forall x \in [a,b]$.

Next let $L_1 = \int_a^c f_1 \left(= \int_a^c f\right)$ &

$L_2 = \int_c^b f_2 \left(= \int_c^b f\right)$

Then $\forall \varepsilon > 0$,

$\exists \delta' > 0$ s.t. $\forall$ tagged partition $\dot{\mathcal{P}}_1$ of $[a,c]$ with $\|\dot{\mathcal{P}}_1\| < \delta'$,

we have $|S(f_1, \dot{\mathcal{P}}_1) - L_1| < \varepsilon/3$

and

$\exists \delta'' > 0$ s.t. $\forall$ tagged partition $\dot{\mathcal{P}}_2$ of $[c,b]$ with $\|\dot{\mathcal{P}}_2\| < \delta''$,

we have $|S(f_2, \dot{\mathcal{P}}_2) - L_2| < \varepsilon/3$.

Now let $\delta_\varepsilon = \min\{\delta', \delta'', \frac{\varepsilon}{6M}\} > 0$ &

<u>Claim</u>: If $\dot{Q}$ is a tagged partition of $[a,b]$ with $\|\dot{Q}\| < \delta_\varepsilon$, then $|S(f;\dot{Q}) - (L_1+L_2)| < \varepsilon$.

If the Claim holds, then $f \in \mathcal{R}[a,b]$ and $\int_a^b f = L_1 + L_2$ and we're done.

<u>Pf of claim</u>

Let $\dot{Q} = \{[x_{i-1}, x_i]; t_i\}_{i=1}^n$

then $x_i - x_{i-1} < \delta_\varepsilon$, $\forall i = 1, \cdots, n$.

<u>Case (i)</u> $c = x_k$ for some $k = 1, \cdots, n-1$. (excluding $x_0 = a$ & $x_n = b$)

Then $\dot{Q} = \{[x_{i-1}, x_i]; t_i\}_{i=1}^k \cup \{[x_{i-1}, x_i]; t_i\}_{i=k+1}^n$

Note that

$\dot{Q}_1 = \{[x_{i-1}, x_i]; t_i\}_{i=1}^k$ is a tagged partition of $[a,c]$ &

$\dot{Q}_2 = \{[x_{i-1}, x_i]; t_i\}_{i=k+1}^n$ is a tagged partition of $[c,b]$

Hence $S(f;\dot{Q}) = S(f_1;\dot{Q}_1) + S(f_2;\dot{Q}_2)$

Since $\|\dot{Q}_1\| \le \|\dot{Q}\| < \delta_\varepsilon \le \delta'$ &

$\|\dot{Q}_2\| \le \|\dot{Q}\| < \delta_\varepsilon \le \delta''$, we have $\quad |S(f_1; \dot{Q}_1) - L_1| < \varepsilon/3$

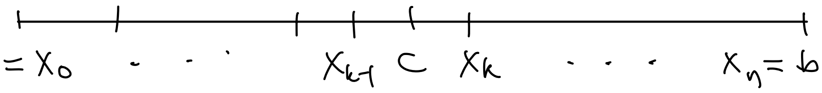
$$|S(f_2; \dot{Q}_2) - L_2| < \varepsilon/3 .$$

Hence $\quad |S(f; \dot{Q}) - (L_1 + L_2)|$

$$\leq |S(f_1; \dot{Q}_1) - L_1| + |S(f_2; \dot{Q}_2) - L_2|$$

$$< \frac{2\varepsilon}{3} < \varepsilon .$$

<u>Case (ii)</u> $\quad c \in (x_{k-1}, x_k)$ for some $k = 1, 2, \cdots, n$.



Then $\quad [x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{k-2}, x_{k-1}] \cup [x_{k-1}, c]$ ← (new subinterval)

with tags $\quad t_1, \quad t_2, \quad \cdots, \quad t_{k-1}, \quad$ & $\quad c$ ← (new tag)

is a tagged partition $\dot{Q}_1$ of $[a, c]$.

Similarly, $\quad$ (new subinterval) → $[c, x_k] \cup [x_k, x_{k+1}] \cup \cdots \cup [x_{n-1}, x_n]$

with tags $\quad c, \quad t_{k+1}, \quad \cdots, \quad t_n$

(new tag)

is a tagged partition $\dot{Q}_2$ of $[c, b]$.

Then

$$S(f; \dot{Q}) = \sum_{i=1}^{k-1} f(t_i)(x_i - x_{i-1}) + f(t_k)(x_k - x_{k-1}) + \sum_{i=k+1}^{n} f(t_i)(x_i - x_{i-1})$$

$$= \left[\sum_{i=1}^{k-1} f(t_i)(x_i - x_{i-1}) + f(c)(c - x_{k-1}) \right] - f(c)(c - x_{k-1})$$

$$+ f(t_k)(x_k - x_{k-1})$$

$$+ \left[f(c)(x_k - c) + \sum_{i=k+1}^{n} f(t_i)(x_i - x_{i-1}) \right] - f(c)(x_k - c)$$

$$= S(f_1, \dot{Q}_1) + S(f_2; \dot{Q}_2) + (f(t_k) - f(c))(x_k - x_{k-1})$$

$$\Rightarrow \quad |S(f; \dot{Q}) - S(f_1, \dot{Q}_1) - S(f_2; \dot{Q}_2)|$$

$$\leq |f(t_k) - f(c)| \, |x_k - x_{k-1}|$$

$$\leq 2M \|\dot{Q}\| < 2M \cdot \frac{\varepsilon}{6M}$$

$$< \frac{\varepsilon}{3} \qquad \text{———} \quad (*)_1$$

Also $\|\dot{Q}_1\| \leq \|\dot{Q}\|$ $\quad$ (as $0 < c - x_{k-1} < x_k - x_{k-1} \leq \|\dot{Q}\|$)

$\therefore \quad \|\dot{Q}_1\| < \delta_\varepsilon < \delta'$

$$\Rightarrow \quad |S(f_1, \dot{Q}_1) - L_1| < \varepsilon/3 . \qquad \text{———} \quad (*)_2$$

Similarly $\quad \|\dot{Q}_2\| \leq \|\dot{Q}\| < \delta_\varepsilon < \delta''$

$$\Rightarrow \quad |S(f_2, \dot{Q}_2) - L_2| < \varepsilon/3 . \qquad \text{———} \quad (*)_3$$

Then by $(*)_1$, $(*)_2$, & $(*)_3$

$$|S(f; \dot{Q}) - (L_1 + L_2)|$$

$$\leq \left| \mathcal{S}(f; \mathring{Q}) - \mathcal{S}(f_1; \mathring{Q}_1) - \mathcal{S}(f_2; \mathring{Q}_2) \right|$$

$$+ \left| \mathcal{S}(f_1; \mathring{Q}_1) - L_1 \right| + \left| \mathcal{S}(f_2; \mathring{Q}_2) - L_2 \right|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

This completes the proof of the Claim & hence the proof of the Thm. ///