(c) $h(x) = x$ (for $x \in [0,1]$) $\in \mathcal{R}[0,1]$ & $\int_0^1 h = \frac{1}{2}$.

Pf: Let $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of I.

Take tags $t_i = q_i$ be the mid-points,

i.e. $$q_i = \frac{x_{i-1} + x_i}{2}.$$

Then the corresponding tagged partition $\dot{Q} = \{[x_{i-1}, x_i]; q_i\}_{i=1}^n$ has Riemann sum

$$S(h; \dot{Q}) = \sum_{i=1}^n h(q_i)(x_i - x_{i-1}) = \sum_{i=1}^n q_i (x_i - x_{i-1})$$

$$= \sum_{i=1}^n \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1}) = \frac{1}{2} \sum_{i=1}^n (x_i^2 - x_{i-1}^2)$$

$$= \frac{1}{2}\left[(x_1^2 - x_0^2) + (x_2^2 - x_1^2) + \cdots + (x_n^2 - x_{n-1}^2)\right]$$

$$= \frac{1}{2}(x_n^2 - x_0^2) = \frac{1}{2} \qquad (x_n = 1, \ x_0 = 0)$$

Now if $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ is a tagged partition with the same partition but arbitrary tags t_i, then $\|\dot{\mathcal{P}}\| = \|\dot{Q}\| < \delta$, and

$$|S(h; \dot{\mathcal{P}}) - S(h, \dot{Q})| = \left| \sum_{i=1}^n h(t_i)(x_i - x_{i-1}) - \sum_{i=1}^n h(q_i)(x_i - x_{i-1}) \right|$$

$$= \left| \sum_{i=1}^n (t_i - q_i)(x_i - x_{i-1}) \right|$$

$$\le \sum_{i=1}^{n} |t_i - q_i|(x_i - x_{i-1}) < \delta \sum_{i=1}^{n} (x_i - x_{i-1})$$

$$= \delta \qquad \left(\text{since } t_i, q_i \in [x_{i-1}, x_i] \text{ and } x_i - x_{i-1} < \delta \right)$$

Using $S(h;\dot{Q}) = \frac{1}{2}$ for any partition with mid-pt tags, we have $\forall\, \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta$,

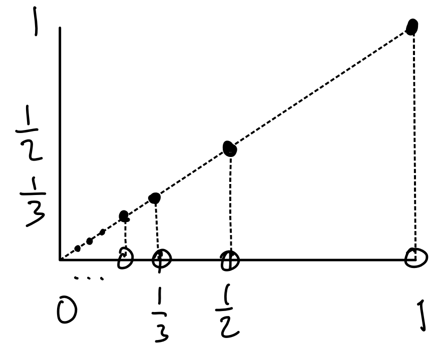
$$|S(h;\dot{\mathcal{P}}) - \tfrac{1}{2}| < \delta .$$

Hence $\forall\, \varepsilon > 0$, take $\delta_\varepsilon = \varepsilon > 0$, we have

$$\dot{\mathcal{P}} \text{ with } \|\dot{\mathcal{P}}\| < \delta_\varepsilon \Rightarrow |S(h;\dot{\mathcal{P}}) - \tfrac{1}{2}| < \varepsilon$$

$\therefore\ h \in \mathcal{R}[0,1]$ & $\int_a^b h = \frac{1}{2}$.

(d) $G(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{1}{n} \ (n = 1, 2, \cdots) \\ 0, & \text{elsewhere on } [0,1] \end{cases}$



is (Riemann) integrable on $[0,1]$

and $\int_0^1 G = 0$.

Pf: $\forall\, \varepsilon > 0$, $E_\varepsilon = \{ x \in [0,1] : G(x) \ge \varepsilon \}$

$= \{ 1, \frac{1}{2}, \cdots, \frac{1}{N_\varepsilon} \}$ where $N_\varepsilon = [\frac{1}{\varepsilon}]$ the largest integer $\le \frac{1}{\varepsilon}$.

is a finite set.

Let $\delta = \dfrac{\varepsilon}{2N_\varepsilon} > 0$.

If $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ is a tagged partition with $\|\dot{\mathcal{P}}\| < \delta$.

Then $$S(G;\dot{\mathcal{P}}) = \sum_{i=1}^{n} G(t_i)(x_i - x_{i-1})$$

$$= \sum_{\substack{i=1 \\ t_i \notin E_\varepsilon}}^{n} G(t_i)(x_i - x_{i-1}) + \sum_{\substack{i=1 \\ t_i \in E_\varepsilon}}^{n} G(t_i)(x_i - x_{i-1})$$

$t_i \notin E_\varepsilon \Rightarrow 0 \le G(t_i) < \varepsilon$

$$\therefore\ 0 \le \sum_{\substack{i=1 \\ t_i \notin E_\varepsilon}}^{n} G(t_i)(x_i - x_{i-1}) < \varepsilon \sum_{i=1}^{n} (x_i - x_{i-1}) = \varepsilon$$

There are only N_ε number of pts in E_ε, & $0 \le G(x) \le 1$, and a tag belongs to at most two subintervals

$$\therefore\ 0 \le \sum_{\substack{i=1 \\ t_i \in E_\varepsilon}}^{n} G(t_i)(x_i - x_{i-1}) < \sum_{\substack{i=1 \\ t_i \in E_\varepsilon}}^{n} \delta < 2N_\varepsilon \cdot \delta = \varepsilon$$

$\leftarrow$ at most $2N_\varepsilon$ terms

Hence

$$0 \le S(G;\dot{\mathcal{P}}) < \varepsilon + \varepsilon = 2\varepsilon, \text{ for any } \dot{\mathcal{P}} \text{ with } \|\dot{\mathcal{P}}\| < \delta_\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$G \in \mathcal{R}[0,1] \quad \text{and} \quad \int_0^1 G = 0.$$ ⌗

Properties of Integral

Thm 7.1.5 Suppose $f, g \in \mathcal{R}[a,b]$. Then

(a) $kf \in \mathcal{R}[a,b]$, $\forall k \in \mathbb{R}$ and

$$\int_a^b kf = k\int_a^b f$$

(b) $f+g \in \mathcal{R}[a,b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

(c) $f(x) \le g(x)$ $\forall x \in [a,b] \Rightarrow \int_a^b f \le \int_a^b g$.

Pf: (a) Ex. (Similar to the proof of (b) & easier)

(b) $f, g \in \mathcal{R}[a,b] \Rightarrow$

$\forall \varepsilon > 0$, $\exists\ \delta_1 > 0$ s.t. $\left|S(f, \dot{\mathcal{P}}) - \int_a^b f\right| < \varepsilon$, $\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta_1$

& $\exists\ \delta_2 > 0$ s.t. $\left|S(g, \dot{\mathcal{P}}) - \int_a^b g\right| < \varepsilon$, $\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta_2$.

Also note that for any $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$,

$$\begin{aligned} S(f+g; \dot{\mathcal{P}}) &= \sum_{i=1}^n (f+g)(t_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) + \sum_{i=1}^n g(t_i)(x_i - x_{i-1}) \\ &= S(f; \dot{\mathcal{P}}) + S(g; \dot{\mathcal{P}}) \end{aligned}$$

Then $\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta = \min\{\delta_1, \delta_2\}^{(70)}$, we have

$$\left| S(f+g; \dot{\mathcal{P}}) - \left(\int_a^b f + \int_a^b g \right) \right|$$

$$\leq \left| S(f; \dot{\mathcal{P}}) - \int_a^b f \right| + \left| S(g; \dot{\mathcal{P}}) - \int_a^b g \right|$$

$$< \varepsilon + \varepsilon = 2\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we've proved that

$$f+g \in \mathcal{R}[a,b] \text{ and } \int_a^b (f+g) = \int_a^b f + \int_a^b g .$$

(c) As in (b), we conclude, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

for $\dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta$,

$$\left| S(f; \dot{\mathcal{P}}) - \int_a^b f \right| < \varepsilon \text{ and } \left| S(g; \dot{\mathcal{P}}) - \int_a^b g \right| < \varepsilon$$

$$\Rightarrow \quad \int_a^b f - \varepsilon < S(f; \dot{\mathcal{P}}) \quad \& \quad S(g; \dot{\mathcal{P}}) < \int_a^b g + \varepsilon .$$

Now $f(x) \leq g(x)$, $\forall x \in [a,b] \Rightarrow$

$$S(f; \dot{\mathcal{P}}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) \leq \sum_{i=1}^n g(t_i)(x_i - x_{i-1}) = S(g; \dot{\mathcal{P}})$$

$$\therefore \int_a^b f - \varepsilon < S(f; \dot{\mathcal{P}}) \leq S(g; \dot{\mathcal{P}}) < \int_a^b g + \varepsilon$$

or $\int_a^b f < \int_a^b g + 2\varepsilon$.

Since $\varepsilon > 0$ is arbitrary, $\int_a^b f \leq \int_a^b g$ ※

Boundedness Theorem

$$\boxed{\underline{\text{Thm 7.1.6}} \quad f \in \mathcal{R}[a,b] \Rightarrow f \text{ is bounded on } [a,b].}$$

(of course, $f \in \mathcal{R}[a,b] \not\Leftarrow f$ is bounded on $[a,b]$, see later section)

Pf Let $f \in \mathcal{R}[a,b]$ and $\int_a^b f = L$.

And suppose on the contrary that f is <u>unbounded</u> on $[a,b]$

$f \in \mathcal{R}[a,b]$ with $\int_a^b f = L$ (Take $\varepsilon = 1$ in the def)

$\Rightarrow \exists \delta > 0$ such that

$\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta$,

$$|S(f;\dot{\mathcal{P}}) - L| < 1$$

$$\Rightarrow \quad |S(f;\dot{\mathcal{P}})| < |L| + 1 \quad \text{———} \quad (*)_1$$

If $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of $[a,b]$.

Then f unbounded

$\Rightarrow \exists$ a subinterval $[x_{i_0-1}, x_{i_0}]$ s.t.

f is unbounded on $[x_{i_0-1}, x_{i_0}]$.

Therefore, we can find t_{i_0} such that

$$|f(t_{i_0})(x_{i_0} - x_{i_0-1})| > |L| + 1 + \Big|\sum_{i \neq i_0} f(x_i)(x_i - x_{i-1})\Big| \quad — (*)_2$$

Then the corresponding tagged partition $\dot{\mathcal{P}} = \{[x_{i-1}, x_i]; t_i\}_{i=1}^{n}$,

with tags $\begin{cases} t_i = x_i & \text{for } i \neq i_0 \\ t_{i_0} \end{cases}$

gives Riemann sum

$$S(f; \dot{\mathcal{P}}) = f(t_{i_0})(x_{i_0} - x_{i_0-1}) + \sum_{i \neq i_0} f(x_i)(x_i - x_{i-1})$$

$$\Rightarrow \quad f(t_{i_0})(x_{i_0} - x_{i_0-1}) = S(f; \dot{\mathcal{P}}) - \sum_{i \neq i_0} f(x_i)(x_i - x_{i-1})$$

$$\Rightarrow \quad |f(t_{i_0})(x_{i_0} - x_{i_0-1})| \leq |S(f; \dot{\mathcal{P}})| + \left|\sum_{i \neq i_0} f(x_i)(x_i - x_{i-1})\right|$$

$$\left(\text{by } (*)_1\right) \qquad < |L| + 1 + \left|\sum_{i \neq i_0} f(x_i)(x_i - x_{i-1})\right|$$

which contradicts $(*)_2$

$\therefore$ f must be bounded. ※