Ch 7 The Riemann Integral

§7.1 Riemann Integral

Def: If $I = [a,b]$ is a closed interval, then a partition of I is a finite, ordered set

$$\mathcal{P} = (x_0, x_1, \cdots, x_n)$$

of points in I such that

$$a = x_0 < x_1 < \cdots < x_n = b$$

Note: A partition $\mathcal{P} = (x_0, x_1, \cdots, x_n)$ is used to divide I into (interior) non-overlapping subintervals:

$$I_1 = [x_0, x_1],\ I_2 = [x_1, x_2],\ \cdots,\ I_n = [x_{n-1}, x_n]$$

Hence an alternate notation for $\mathcal{P}$ is

$$\boxed{\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^{n}}$$

Def: The norm (or mesh) of $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^{n}$ is defined by

$$\|\mathcal{P}\| = \max_{i=1,\cdots,n} \{x_i - x_{i-1}\}$$

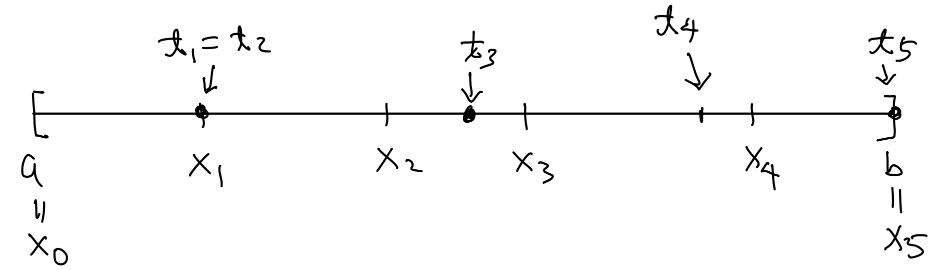
$$= \max\{x_1 - x_0, x_2 - x_1, \cdots, x_n - x_{n-1}\}$$

$$(= \text{length of largest subinterval})$$

<u>Def</u>: (1) If $t_i \in I_i = [x_{i-1}, x_i]$, $\forall i = 1, \cdots, n$ has been selected of each subinterval of a partition $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^{n}$ of $I = [a,b]$, then t_i are called <u>tags</u> of I_i.

(2) The partition $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^{n}$, together with tags t_i is called a <u>tagged partition</u> of $I = [a,b]$ and is denoted by

$$\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$$

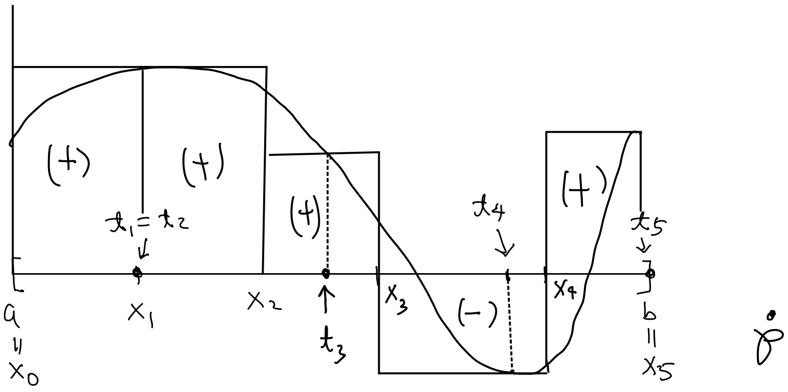


<u>Def</u>: If $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ is a <u>tagged partition</u> of $I = [a,b]$, then the <u>Riemann sum</u> of a function $f : [a,b] \to \mathbb{R}$ is defined by

(may not "continuous")

$$\boxed{S(f; \dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})}$$

<u>Remark</u>: This definition works for the case that $\dot{\mathcal{P}}$ is a <u>subset</u> of a partition, and not the entire partition.



$S(f;\dot{\mathcal{P}})$ = sum of (signed) areas of the n rectangles with bases $[x_{i-1}, x_i]$ & heights $f(t_i)$, $i=1,\cdots,n$.

<u>Def 7.1.1</u> (1) A function $f:[a,b]\to\mathbb{R}$ is said to be <u>Riemann integrable</u> on $[a,b]$ if

$\exists\, L\in\mathbb{R}$ such that

$\forall\, \varepsilon>0$, $\exists\, \delta_\varepsilon>0$ such that

$\forall$ tagged partition $\dot{\mathcal{P}}$ of $[a,b]$ with $\|\dot{\mathcal{P}}\|<\delta_\varepsilon$,

$$|S(f;\dot{\mathcal{P}})-L|<\varepsilon.$$

(2) The <u>set of all Riemann integrable functions on $[a,b]$</u> will be denoted by <u>$\mathcal{R}[a,b]$</u>.

(3) If $f\in\mathcal{R}[a,b]$, the number L is <u>uniquely determined</u> (Thm 7.1.2), called the <u>Riemann integral of f over $[a,b]$</u>, & is denoted by

$$\int_a^b f \quad \text{or} \quad \int_a^b f(x)\,dx$$

(x is a dummy variable, can be replaced by any other notation)

Remark: One often says that L is "the limit" of $S(f; \dot{\mathcal{P}})$ as $\|\dot{\mathcal{P}}\| \to 0$. However $S(f; \dot{\mathcal{P}})$ is not a function of $\|\dot{\mathcal{P}}\|$, it is not the limit (of functions) that defined before.

(there are many $\dot{\mathcal{P}}$ with same $\|\dot{\mathcal{P}}\|$)

$\boxed{\text{Thm 7.1.2 If } f \in \mathcal{R}[a,b]\text{, then the value of the integral is uniquely determined.}}$

Pf: Suppose L' and L'' both satisfy the definition 7.1.1.

Then $\forall \varepsilon > 0$, $\exists\ \delta'_{\frac{\varepsilon}{2}} > 0$ such that

$$|S(f; \dot{\mathcal{P}}_1) - L'| < \frac{\varepsilon}{2} \quad \forall\ \dot{\mathcal{P}}_1 \text{ with } \|\dot{\mathcal{P}}_1\| < \delta'_{\frac{\varepsilon}{2}}$$

and $\exists\ \delta''_{\frac{\varepsilon}{2}} > 0$ such that

$$|S(f; \dot{\mathcal{P}}_2) - L''| < \frac{\varepsilon}{2} \quad \forall\ \dot{\mathcal{P}}_2 \text{ with } \|\dot{\mathcal{P}}_2\| < \delta''_{\frac{\varepsilon}{2}}.$$

Let $\delta_\varepsilon = \min\{\delta'_{\frac{\varepsilon}{2}}, \delta''_{\frac{\varepsilon}{2}}\} > 0$.

If $\dot{\mathcal{P}}$ is a tagged partition with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$,

then $\|\dot{\mathcal{P}}\| < \delta'_{\varepsilon/2}$ and $\|\dot{\mathcal{P}}\| < \delta''_{\varepsilon/2}$.

Hence $|S(f; \dot{\mathcal{P}}) - L'| < \frac{\varepsilon}{2}$ and $|S(f; \dot{\mathcal{P}}) - L''| < \frac{\varepsilon}{2}$.

$$\Rightarrow |L' - L''| \le |S(f;\dot{\mathcal{P}}) - L'| + |S(f;\dot{\mathcal{P}}) - L''|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $L' = L''$. #

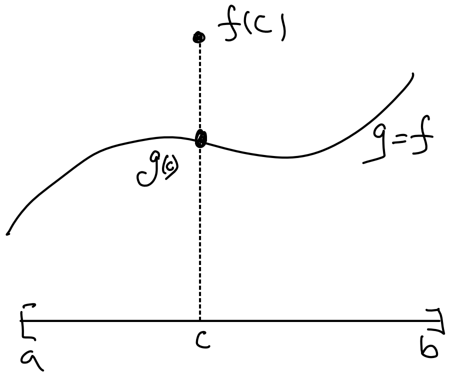
<u>Thm 7.1.3</u> If
- $g \in \mathcal{R}[a,b]$ (Riemann integrable)
- $f(x) = g(x)$ except for a <u>finite number of points</u>.

Then
- $f \in \mathcal{R}[a,b]$ and
- $\int_a^b f = \int_a^b g$.

<u>Pf</u>: Only need to prove the case that

$f(x) = g(x)$ except for <u>one point</u> in $[a,b]$.

Then induction implies the Theorem.



Let c be the point in $[a,b]$

s.t. $f(c) \neq g(c)$.

Then $f(x) = g(x)$, $\forall x \in [a,b] \setminus \{c\}$.

Let $L = \int_a^b g$. (By assumption that $g \in \mathcal{R}[a,b]$, it exists)

For any tagged partition $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$,

Then (i) $c \in (x_{i_0-1}, x_{i_0})$ for some $i_0 \in \{1, 2, \cdots, n\}$

or (ii) $c = x_{i_0}$ for some $i_0 \in \{1, 2, \cdots, n\}$.

For case (i), $f(x) = g(x)$ for all $[x_{i-1}, x_i]$, $i \neq i_0$

$$\Rightarrow f(t_i) = g(t_i)$$

And hence

$$S(f; \dot{\mathcal{P}}) - S(g; \dot{\mathcal{P}}) = \cancel{\sum_{i \neq i_0} f(t_i)(x_i - x_{i-1})} - \cancel{\sum_{i \neq i_0} g(t_i)(x_i - x_{i-1})}$$

$$+ f(t_{i_0})(x_{i_0} - x_{i_0-1}) - g(t_{i_0})(x_{i_0} - x_{i_0-1})$$

$$= \big(f(t_{i_0}) - g(t_{i_0})\big)(x_{i_0} - x_{i_0-1})$$

$$\Rightarrow |S(f; \dot{\mathcal{P}}) - S(g; \dot{\mathcal{P}})| \leq |f(t_{i_0}) - g(t_{i_0})|\, |x_{i_0} - x_{i_0-1}|$$

$$\leq \big(|f(c)| + |g(c)|\big) \|\dot{\mathcal{P}}\|.$$

Similarly for case (ii)

$$S(f; \dot{\mathcal{P}}) - S(g; \dot{\mathcal{P}}) = \sum_{\substack{i \neq i_0 \\ i_0+1}} \cancelto{0}{[f(t_i) - g(t_i)]}(x_i - x_{i-1})$$

$$+ f(t_{i_0})(x_{i_0} - x_{i_0-1}) - g(t_{i_0})(x_{i_0} - x_{i_0-1})$$

$$+ f(t_{i_0+1})(x_{i_0+1} - x_{i_0}) - g(t_{i_0+1})(x_{i_0+1} - x_{i_0})$$

$$= \big(f(t_{i_0}) - g(t_{i_0})\big)(x_{i_0} - x_{i_0-1}) + \big(f(t_{i_0+1}) - g(t_{i_0+1})\big)(x_{i_0+1} - x_{i_0})$$

$\therefore \left| S(f;\dot{\mathcal{P}}) - S(g;\dot{\mathcal{P}}) \right| \leq (|f(c)|+|g(c)|)\,\|\dot{\mathcal{P}}\| + (|f(c)|+|g(c)|)\,\|\dot{\mathcal{P}}\|$

$$= 2(|f(c)|+|g(c)|)\,\|\dot{\mathcal{P}}\| .$$

Hence, in both cases,

$$|S(f;\dot{\mathcal{P}}) - S(g;\dot{\mathcal{P}})| \leq 2(|f(c)|+|g(c)|)\,\|\dot{\mathcal{P}}\|$$

Therefore, $\forall \varepsilon > 0$, for $\delta_1 = \dfrac{\varepsilon}{5(|f(c)|+|g(c)|)}$, we have

$\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta_1$,

$$|S(f;\dot{\mathcal{P}}) - S(g;\dot{\mathcal{P}})| \leq 2(|f(c)|+|g(c)|) \cdot \frac{\varepsilon}{5(|f(c)|+|g(c)|)} < \frac{\varepsilon}{2} .$$

Now, by $g \in \mathcal{R}[a,b]$ & $L = \int_a^b g$, $\exists \delta_2 > 0$ s.t.

$\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta_2$,

$$|S(g;\dot{\mathcal{P}}) - L| < \frac{\varepsilon}{2} .$$

Letting $\delta = \min\{\delta_1, \delta_2\} > 0$, we have

$\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta$,

$$|S(f;\dot{\mathcal{P}}) - L| \leq |S(f;\dot{\mathcal{P}}) - S(g;\dot{\mathcal{P}})| + |S(g;\dot{\mathcal{P}}) - L|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore f \in \mathcal{R}[a,b]$ and $\int_a^b f = L = \int_a^b g$. ※

Eg 7.1.4

(a) If $f \equiv$ const., then $f \in \mathcal{R}[a,b]$

Pf: Let the const. be k.

Then $f(x) = k \quad \forall x \in [a,b]$

If $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ be a tagged partition of $[a,b]$,

then corresponding Riemann sum

$$S(f;\dot{\mathcal{P}}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$$

$$= \sum_{i=1}^n k(x_i - x_{i-1}) = k(b-a)$$

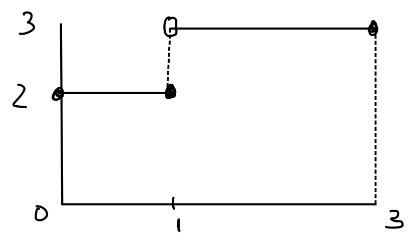
$\therefore \forall \varepsilon > 0$, we can just pick any $\delta > 0$ and have

$$|S(f;\dot{\mathcal{P}}) - k(b-a)| = 0 < \varepsilon, \quad \forall \dot{\mathcal{P}} \text{ with } \|\dot{\mathcal{P}}\| < \delta$$

$\therefore \quad f \equiv k \in \mathcal{R}[a,b]$.

In fact, we have proved that $\int_a^b k = k(b-a)$ $\divideontimes$

(b) $g: [0,3] \to \mathbb{R}$ defined by $g(x) = \begin{cases} 3, & 1 < x \le 3 \\ 2, & 0 \le x \le 1 \end{cases}$



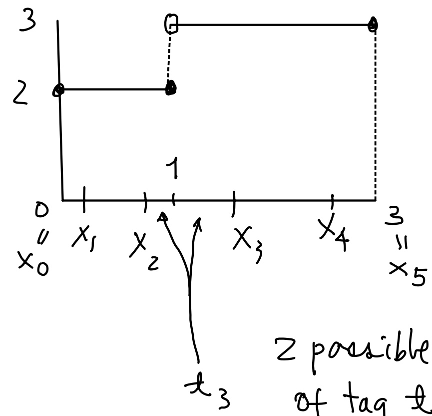
is (Riemann) integrable & $\int_0^3 g = 8$.

Pf: Let $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$

Let $k = 1, \cdots, n$ such that

$0 \le t_1 \le \cdots \le t_k \le 1$ and

$1 < t_{k+1} \le \cdots \le t_n \le 3$



2 possible situations of tag $t_k \gtrless 1$

Let $\dot{\mathcal{P}}_1 = \{[x_{i-1}, x_i], t_i\}_{i=1}^{k}$

and $\dot{\mathcal{P}}_2 = \{[x_{i-1}, x_i], t_i\}_{i=k+1}^{n}$

(Using the remark of the definition of Riemann sum)

we have

$$S(g; \dot{\mathcal{P}}) = \sum_{i=1}^{k} g(t_i)(x_i - x_{i-1}) + \sum_{i=k+1}^{n} g(t_i)(x_i - x_{i-1})$$

$$= S(g; \dot{\mathcal{P}}_1) + S(g; \dot{\mathcal{P}}_2)$$

Suppose that $\|\dot{\mathcal{P}}\| < \delta$ for some $\delta > 0$.

Then $t_k \le 1$, $x_{k-1} \le t_k \le x_k$ and $x_k - x_{k-1} < \delta$,

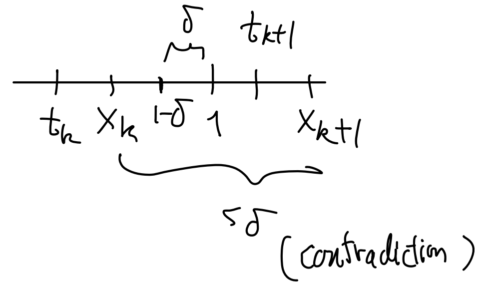
we have $\quad x_k < \delta + x_{k-1} \le \delta + t_k \le 1 + \delta$

$\therefore$ (notation in Textbook) $\quad U_1 = \bigcup_{i=1}^{k} [x_{i-1}, x_i] = [0, x_k] \subset [0, 1+\delta]$.

On the other hand, we claim that $1 - \delta \le x_k$.

Suppose not, then $1 - \delta > x_k$.

From the choice of k, $t_{k+1} > 1$.



$\therefore \quad x_{k+1} \geq t_{k+1} > 1$

Hence $\quad \delta > x_{k+1} - x_k > 1 - (1-\delta) = \delta$,

which is a contradiction.

$$\therefore \qquad 1-\delta \leq x_k .$$

Together we have

$$[0, 1-\delta] \subset U_1 = \bigcup_{i=1}^{k} [x_{i-1}, x_i] = [0, x_k] \subset [0, 1+\delta]$$

Therefore

$$S(g; \dot{P}_1) = \sum_{i=1}^{k} g(t_i)(x_i - x_{i-1}) = 2x_k$$

$$(t_i \leq 1 \Rightarrow g(t_i) = 2)$$

$$\Rightarrow \qquad 2(1-\delta) \leq S(g; \dot{P}_1) \leq 2(1+\delta) \quad \text{——} \quad (*)_1 .$$

Similarly,

$$S(g; \dot{P}_2) = \sum_{i=k+1}^{n} g(t_i)(x_i - x_{i-1}) = 3(3 - x_k)$$

$$(t_i > 1 \Rightarrow g(t_i) = 3)$$

$$\Rightarrow \quad 3(3-(1+\delta)) \leq S(g; \dot{P}_2) \leq 3(3-(1-\delta))$$

$$3(2-\delta) \leq S(g; \dot{P}_2) \leq 3(2+\delta) \quad \text{——} \quad (*)_2$$

By $(*)_1$ + $(*)_2$, we have, for $\dot{P}$ satisfying $\|\dot{P}\| < \delta$,

$$2(1-\delta)+3(2-\delta) \leq S(g;\mathring{P}) \leq 2(1+\delta)+3(2+\delta)$$

i.e. $$8-5\delta \leq S(g;\mathring{P}) \leq 8+5\delta$$

$$\therefore \quad |S(g;\mathring{P})-8| \leq 5\delta .$$

Therefore $\forall \varepsilon>0$, we can take $\delta_\varepsilon = \frac{\varepsilon}{10} > 0$ to have

$\forall \mathring{P}$ with $\|\mathring{P}\| < \delta_\varepsilon$,

$$|S(g;\mathring{P})-8| \leq 5 \cdot \frac{\varepsilon}{10} < \varepsilon .$$

※