(Cont'd)

($\Leftarrow$) Assuming $f''(x) \geq 0$, $\forall x \in I$.

$\forall t \in [0,1]$ & $\forall x_1, x_2 \in I$,

let $x_0 = (1-t)x_1 + t x_2$ ( clearly $\in I$ )

Then Taylor's Thm $\Rightarrow$

$$f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + \frac{1}{2} f''(c_1)(x_1 - x_0)^2$$

$$\geq f(x_0) + f'(x_0)(x_1 - x_0)$$

(for some $c_1$ between $x_1$ & $x_0$)

and $$f(x_2) = f(x_0) + f'(x_0)(x_2 - x_0) + \frac{1}{2} f''(c_2)(x_2 - x_0)^2$$

$$\geq f(x_0) + f'(x_0)(x_2 - x_0)$$

(for some $c_2$ between $x_2$ & $x_0$)

$(1-t)f(x_1) + t f(x_2)$

$$\geq (1-t)f(x_0) + t f(x_0) + f'(x_0)\left[(1-t)(x_1 - x_0) + t(x_2 - x_0)\right]$$

$$= f(x_0) + f'(x_0)\left[(1-t)x_1 + t x_2 - x_0\right]$$

$$= f((1-t)x_1 + t x_2),$$ since $x_0 = (1-t)x_1 + t x_2$

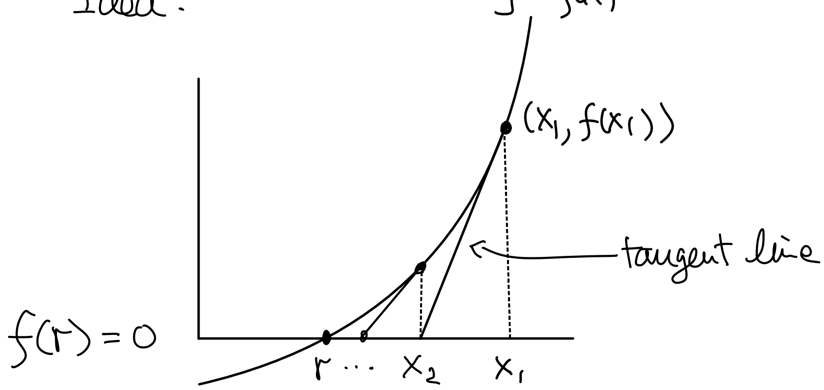
$\#$

# Newton's Method

Goal:

Find root $r$ of $f$

Idea:




Equation of tangent line: $\quad y - f(x_1) = f'(x_1)(x - x_1)$

$\therefore$ its intersection with x-axis, $x_2$, satisfies

$$-f(x_1) = f'(x_1)(x_2 - x_1)$$

$$\therefore \qquad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad (\text{provided } f'(x_1) \neq 0)$$

Successively $\quad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} \qquad \left(\begin{array}{c}\text{provided } f'(x_k) \neq 0 \\ \text{for } k = 1, \cdots, n\end{array}\right)$

Hope (to find condition such that)

$$x_n \to r \qquad (\text{a zero (root) of } f).$$

Thm 6.4.7 (Newton's Method)

Let
- $f: [a,b] \to \mathbb{R}$ twice differentiable $\quad (a<b)$
- $f(a)f(b) < 0$ (ie $f(a)$, $f(b)$ have opposite signs)
- $\exists$ constants $m>0$, $M \geq 0$ such that

$$|f'(x)| \geq m > 0 \quad \& \quad |f''(x)| \leq M, \quad \forall x \in [a,b].$$

Then $\exists$ a subinterval $I^* \subset [a,b]$

- containing a zero $r$ of $f$ such that
- $\forall x_1 \in I^*$, the sequence $(x_n)$ defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \forall n = 1,2,3 \cdots$$

belongs to $I^*$ and

- $\lim_{n\to\infty} x_n = r$

Moreover
- $|x_{n+1} - r| \leq K |x_n - r|^2 \quad \forall n = 1,2,3 \cdots$

where $K = M/2m$.

Pf: Since $f(a)f(b)<0$, $f(a)$, $f(b)$ have opposite signs (& nonzero)

$f$ twice differentiable $\Rightarrow$ $f$ cts. on $[a,b]$.

Intermediate Thm $\Rightarrow \exists\, r \in (a,b)$ such that $f(r) = 0$.

Note that $|f'(x)| \geq m > 0, \forall x \in [a,b]$, Rolle's Thm

$\Rightarrow$ $r$ is the unique zero of $f$ in $[a,b]$.

i.e. $f(x) \neq 0, \forall x \in [a,b] \setminus \{r\}$, (Ex!)

Now $\forall x' \in I$, Taylor's Thm $\Rightarrow$

$$0 = f(r) = f(x') + f'(x')(r-x') + \frac{f''(c')}{2}(r-x')^2$$

for some $c'$ between $r$ & $x'$.

(since $f$ is twice diff.)

If $x'' = x' - \dfrac{f(x')}{f'(x')}$, we have

$$x'' = x' + \frac{f'(x')(r-x') + \frac{f''(c')}{2}(r-x')^2}{f'(x')}$$

$$= r + \frac{1}{2}\frac{f''(c')}{f'(x')}(r-x')^2$$

$$\Rightarrow \quad |x''-r| \leq \frac{1}{2}\frac{|f''(c')|}{|f'(x')|}|x'-r|^2$$

$$\leq \frac{1}{2}\frac{M}{m}|x'-r|^2 = K|x'-r|^2. \quad \text{———} \quad (*)$$

Choose $\delta > 0$ such that

$$\delta < 1/K \quad \& \quad [r-\delta, r+\delta] \subset [a,b],$$

and let $I^* = [r-\delta, r+\delta]$

Then, if $x_n \in I^* (\subset [a,b])$ for some $n = 1,2,3,\cdots$, we have, from $(*)$,

$$|x_{n+1}-r| \leq K|x_n-r|^2 \leq K\delta^2 < \delta$$

$\therefore \; x_{n+1} \in I^*.$

i.e. $x_n \in I^* \Rightarrow x_{n+1} \in I^*$.

Therefore, if $x_1 \in I^*$, induction $\Rightarrow$

the sequence $(x_n) \subset I^*$,

and satisfies the required inequality

$$|x_{n+1}-r| \leq K|x_n-r|^2, \; \forall \; n=1,2,3,\dots$$

Finally, to see "limit", we note 1st that

$$|x_{n+1}-r| \leq K|x_n-r|^2 \leq K\delta\,|x_n-r| \qquad (*)_2$$

Then iterate $(*)_2$:

$$|x_{n+1}-r| \leq (K\delta)|x_n-r| \leq (K\delta)\left(K\delta|x_{n-1}-r|\right)$$

$$= (K\delta)^2|x_{n-1}-r| \leq \dots \leq (K\delta)^n|x_1-r|$$

Since $K\delta<1$, $(K\delta)^n \to 0$ as $n\to\infty$,

and $|x_1-r|$ is a constant, we have

$$|x_{n+1}-r| \to 0 \quad \text{as } n\to\infty$$

i.e. $\lim_{n\to\infty} x_n = r$ $\quad$ ##

eg 6.4.8 Using Newton's Method to approximate $\sqrt{2}$.

Soln: Convert the problem to a problem of finding root in order to use Newton's Method:

Consider $f(x) = x^2 - 2 \quad \forall x \in \mathbb{R}$.

Calculation: $f'(x) = 2x$ ($\neq 0$ near the root, as 0 is not a root)

($f''$ exists and satisfies the condition, but we don't need to find it explicitly in the approximation.)

One need to guess an initial point $x_1$.

Since $1^2 = 1$, $2^2 = 4$, ($f(1) = -1$, $f(2) = 2$)

it seems reasonable to try $x_1 = 1$.

Note that $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n}$$

$$= x_n - \frac{1}{2}x_n + \frac{1}{x_n}$$

$$= \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

$\therefore x_1 = 1 \Rightarrow$ $$x_2 = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2} = 1.5$$

$$x_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{17}{12} \simeq 1.416666$$

$$\vdots$$

(Check!) $x_5 \approx 1.41421356237\mathbf{2}$ (correct to 11 places).

Remarks

(a) (*) can be written as $(K|x_{n+1}-r|) \le (K|x_n - r|)^2$

Hence if $K|x_n - r| < 10^{-m}$,

then $K|x_{n+1}-r| < 10^{-2m}$

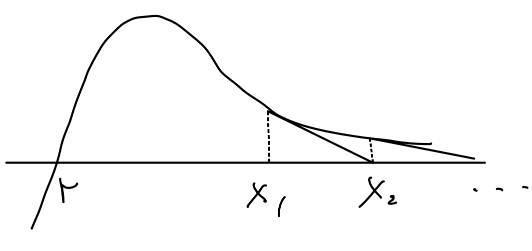
$\therefore$ number of significant digits in $K|x_n - r|$

has been doubled.

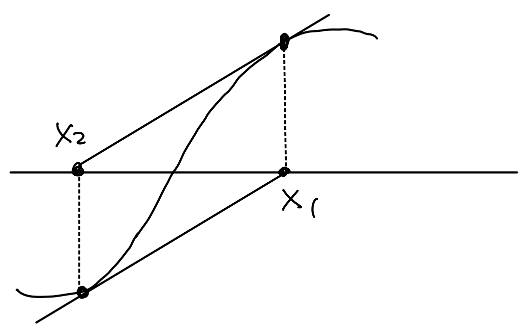
And hence, the sequence $(x_n)$ generated by Newton's method is said to "converge quadratically".

(b) Choose of initial $x_1$ is important (ie. has to be in $I^*$), otherwise $(x_n)$ may not converge to the zero (root).

Possible situations:




$(x_n \to \infty)$




(seq is $(x_1, x_2, x_1, x_2, x_1, x_2, \dots)$)
no limit