Case (b) $L = \pm\infty$. (of Thm 6.3.5)

subcase $L = +\infty$

$$\lim_{x \to a+} \frac{f'(x)}{g'(x)} = +\infty \quad \text{implies that}$$

$$\forall M > 1, \ \exists \delta > 0 \ \text{s.t.} \ \frac{f'(u)}{g'(u)} > M \quad \forall u \in (a, a+\delta)$$

and hence $$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} > M, \quad \forall a < \alpha < \beta < a+\delta$$

As in case (a), $\lim_{x \to a+} g(x) = +\infty$ implies

$\exists$ c & $c_1 \in (a, a+\delta)$ such that

- $a < c_1 < c < a+\delta$
- $g(\alpha) > 0, \ \forall \alpha \in (a, c]$
- $0 < \dfrac{g(c)}{g(\alpha)} < \dfrac{1}{2}, \quad \forall \alpha \in (a, c_1)$
- $0 \leq \dfrac{|f(c)|}{g(\alpha)} < \dfrac{1}{2}, \quad \forall \alpha \in (a, c_1)$

Letting $\beta = c$, we have

$$\frac{f(c) - f(\alpha)}{g(c) - g(\alpha)} > M, \quad \forall \alpha \in (a, c)$$

(both terms < 0)

And hence for $\alpha \in (a, c_1)$

$\left(1 - \frac{g(c)}{g(\alpha)} > \frac{1}{2}\right)$

$$\frac{f(\alpha) - f(c)}{g(\alpha)} > M\left(1 - \frac{g(c)}{g(\alpha)}\right) > \frac{1}{2} M, \quad \forall \alpha \in (a, c_1)$$

$$\Rightarrow \quad \frac{f(\alpha)}{g(\alpha)} > \frac{1}{2}M + \frac{f(c)}{g(\alpha)} > \frac{1}{2}M - \frac{1}{2}$$

$$= \frac{1}{2}(M-1), \qquad \forall \alpha \in (a, c_1)$$

Since $M>1$ is arbitrary, this shows that

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = +\infty .$$

Subcase of "$L=-\infty$" is similar. $\divideontimes$

eg 6.3.6

(a) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x}$

- $f(x) = \ln x$ has derivative $f'(x) = \frac{1}{x}$ on $(0,\infty)$
- $g(x) = x$ has derivative $g'(x) = 1 \neq 0$ on $(0,\infty)$
- $\displaystyle\lim_{x\to\infty} g(x) = +\infty$
- $\displaystyle\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{1/x}{1} = 0$

$\therefore$ L'Hospital Rule II $\Rightarrow$ $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x} = 0$.

(usually, one simply write $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$)

(b) $\lim_{x\to\infty} e^{-x}x^2 = \lim_{x\to\infty} \frac{x^2}{e^x}$

- $(x^2)' = 2x$, $\forall x$
- $(e^x)' = e^x \neq 0$, $\forall x$
- $e^x \to +\infty$ as $x \to +\infty$

But $\lim_{x\to\infty} \frac{2x}{e^x}$ still indeterminate.

So we need to start with $\lim_{x\to\infty} \frac{2x}{e^x}$ first:

- $(2x)' = 2$, $\forall x$
- $(e^x)' = e^x \neq 0$, $\forall x$
- $e^x \to +\infty$ as $x \to +\infty$
- $\lim_{x\to\infty} \frac{2}{e^x} = 0$ (exists)

$\therefore$ L'Hospital Rule $\Rightarrow$ $\lim_{x\to\infty} \frac{2x}{e^x} = 0$ (exists)

And applying L'Hospital Rule again, $\lim_{x\to\infty} \frac{x^2}{e^x} = 0$.

(We usually just write

$$\lim_{x\to\infty} e^{-x}x^2 = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0.)$$

(c) $\lim\limits_{x\to0^+} \dfrac{\ln \sin x}{\ln x} = \lim\limits_{x\to0^+} \dfrac{(\ln\sin x)'}{(\ln x)'}$

$$= \lim_{x\to0^+} \frac{\frac{\cos x}{\sin x}}{\frac{1}{x}} = \lim_{x\to0^+} \cos x \cdot \frac{x}{\sin x}$$

$$= 1 \qquad \left(\text{as } \lim_{x\to0^+} \frac{x}{\sin x} = 1 = \lim_{x\to0^+} \cos x \right)$$

(d) It is easy to see $\lim\limits_{x\to\infty} \dfrac{x-\sin x}{x+\sin x} = \lim\limits_{x\to\infty} \dfrac{1-\frac{\sin x}{x}}{1+\frac{\sin x}{x}} = 1$.

However, $\lim\limits_{x\to\infty} \dfrac{(x-\sin x)'}{(x+\sin x)'} = \lim\limits_{x\to\infty} \dfrac{1-\cos x}{1+\cos x}$ doesn't exist.

$\therefore$ The condition "$\lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)}$ exists" is sufficient, but not <u>necessary</u>

for existence of $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$.