§ 6.3 L'Hospital's Rule

Recall: If $\lim_{x\to c} f(x) = A$

$$\lim_{x\to c} g(x) = B \neq 0,$$

then
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

Question: What can we say about the case that $B = 0$?

(1) If $A \neq 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \infty$ ($\pm$ depends on $\operatorname{sgn}(A)$ & $\operatorname{sgn}(g(x))$ near $x=c$)
(including jumping from $\pm\infty$, i.e. not exist, $\lim_{x\to c} \left|\frac{f(x)}{g(x)}\right| = \infty$)

(2) <u>Indeterminate</u> if $A = 0$:

e.g.

$f(x) = Lx^2$, $g(x) = x^2$: $\lim_{x\to 0} \frac{f(x)}{g(x)} = L$ (finite, $L \neq 0$)

$f(x) = x^3$, $g(x) = x^2$: $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$

$f(x) = x^2$, $g(x) = x^4$: $\lim_{x\to 0} \frac{f(x)}{g(x)} = \infty$

<u>Symbol</u> for this <u>indeterminate form</u>: $0/0$

Other <u>indeterminate forms</u>:

$$\infty/\infty, \quad 0\cdot\infty, \quad 0^0, \quad 1^\infty, \quad \infty^0, \quad \infty-\infty$$

eg: 0^0 denotes indeterminate form of $\lim_{x\to c} f(x)^{g(x)}$

with $\lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x)$.

and $\infty - \infty$ denotes indeterminate form of $\lim_{x\to c} (f(x) - g(x))$

with $\lim_{x\to c} f(x) = +\infty = \lim_{x\to c} g(x)$.

$(\text{or } -\infty)$

Note: Indeterminate forms $0 \cdot \infty$, 0^0, 1^∞, ∞^0 & $\infty - \infty$ can be reduced to the form $0/0$ or ∞/∞ by taking log, exp, or algebraic manipulations.

eg. $\infty - \infty$ $\quad \lim_{x\to c} \left(f(x) - g(x)\right)$ with $\begin{cases} \lim_{x\to c} f(x) = -\infty \\ \lim_{x\to c} g(x) = -\infty \end{cases}$

$$= \lim_{x\to c} \log e^{f(x) - g(x)}$$

$$= \lim_{x\to c} \log \frac{e^{f(x)}}{e^{g(x)}}$$

and one can consider $\lim_{x\to c} \frac{e^{f(x)}}{e^{g(x)}}$ which is of the form $0/0$.

1st Result

Thm 6.3.1 Let
- $f, g : [a,b] \to \mathbb{R}$ $(a<b)$
- $f(a) = g(a) = 0$
- $g(x) \neq 0 \quad \forall\, x \in (a,b)$

If f and g are <u>differentiable at a</u> (1-sided) with <u>$g'(a) \neq 0$</u>, then $\lim\limits_{x\to a^+} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

<u>Remarks:</u>

(1) $f(a) = g(a) = 0$ is necessary!

counterexample: $f(x) = x+17$, $g(x) = 2x+3$ on $[0,1]$.

Then $f(0) = 17 \neq 0$, $g(0) = 3 \neq 0$. (The particular condition: <u>not satisfied</u>)

$f'(0) = 1$, $g'(0) = 2 \neq 0$ (Other conditions: satisfied)

And $\lim\limits_{x\to 0} \frac{f(x)}{g(x)} = \frac{17}{3} \neq \frac{1}{2} = \frac{f'(0)}{g'(0)}$.

(2) No need to assume differentiability (or even continuity) in (a,b).

(3) The Thm holds for the other end point b with

$$\lim_{x\to b^-} \frac{f(x)}{g(x)} = \frac{f'(b)}{g'(b)} \quad \text{provided} \quad \begin{cases} f'(b) \,\&\, g'(b) \text{ exist (1-sided)} \\ f(b)=g(b)=0 \,\&\, g'(b)\neq 0; \end{cases}$$

and also interior point $c\in(a,b)$ with

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \quad \text{provided} \quad \begin{cases} f'(c) \,\&\, g'(c) \text{ exist } \,\&\, g'(c)\neq 0. \\ f(c)=g(c)=0, \end{cases}$$

Pf: By $f(a)=g(a)=0$, & $g(x)\neq 0$ $\forall x\in(a,b)$

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \left(\frac{f(x)-f(a)}{x-a}\right)\Big/\left(\frac{g(x)-g(a)}{x-a}\right), \quad \forall x\in(a,b)$$

$$\therefore \lim_{x\to a^+} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \quad \text{as} \quad \left.\begin{matrix} f'(a)=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}; \\ g'(a)=\lim_{x\to a^+}\frac{g(x)-g(a)}{x-a}\neq 0 \end{matrix}\right\} \text{exist}$$

※

eg: Thm 6.3.1 can be applied as follow (interior point):

$$\lim_{x\to 0} \frac{x^2+x}{\sin 2x} = \frac{\frac{d}{dx}(x^2+x)\big|_{x=0}}{\frac{d}{dx}\sin 2x\big|_{x=0}} = \frac{1}{2}.$$

For further results, we need

Thm 6.3.2 (Cauchy Mean Value Theorem)

Let
- $f, g : [a,b] \to \mathbb{R}$ continuous $(a<b)$
- f, g differentiable on (a,b)
- $g'(x) \neq 0$, $\forall x \in (a,b)$

Then $\exists\ c \in (a,b)$ s.t. $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$.

Remarks: (1) One may tempted to think of the following wrong proof:

$$\text{MVT} \Rightarrow \exists c \text{ s.t. } f(b)-f(a) = f'(c)(b-a)$$

$$\text{and } g(b)-g(a) = g'(c)(b-a)$$

Hence $$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

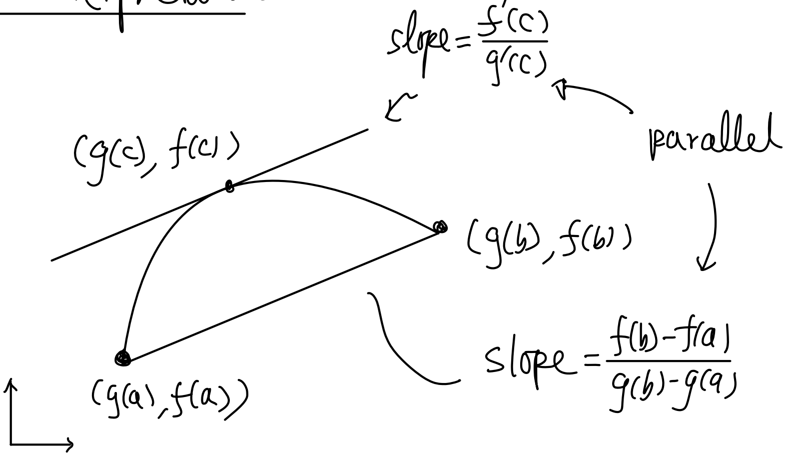
The mistake is that the "c" given by the MVT depends on the functions f & g. Careful notations should be

$$\exists\ c_f \text{ s.t. } f(b)-f(a) = f'(c_f)(b-a) \quad \&$$

$$\exists\ c_g \text{ s.t. } g(b)-g(a) = g'(c_g)(b-a).$$

But c_f may not equal c_g.

(2) Geometric interpretation



(3) Clearly, if $g(x) = x$, Cauchy MVT reduces to MVT.

Pf (of Cauchy MVT).

Since $g'(x) \neq 0$, $\forall x \in (a,b)$, we have $g(b) \neq g(a)$.

Otherwise the function $g(x) - g(a)$ satisfies $\begin{cases} g(b) - g(a) = 0 \\ g(a) - g(a) = 0 \end{cases}$

and Rolle's Thm $\Rightarrow \exists\, c \in (a,b)$ s.t. $g'(c) = (g(x) - g(a))'\big|_{x=c} = 0$

contradiction.

Hence we can define

$$h(x) = \frac{f(b)-f(a)}{g(b)-g(a)}\big(g(x)-g(a)\big) - \big(f(x)-f(a)\big), \quad \forall x \in [a,b]$$

Clearly, h is continuous on $[a,b]$ & differentiable on (a,b)

(by the assumption on f & g). Moreover,

$$h(b) = \frac{f(b)-f(a)}{g(b)-g(a)}\big(g(b)-g(a)\big) - \big(f(b)-f(a)\big) = 0 \quad \text{and}$$

$$h(a) = \frac{f(b)-f(a)}{g(b)-g(a)}\left(g(a)-g(a)\right) - \left(f(a)-f(a)\right) = 0$$

$\therefore$ Rolle's Thm $\Rightarrow \exists\, c \in (a,b)$ s.t.

$$0 = h'(c) = \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) - f'(c)$$

Since $g'(c) \neq 0$, we have $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$

$\divideontimes$

L'Hospital's Rule I

Remarks:

(1) No need to assume $f'(a)$, $g'(a)$ exist as in Thm 6.3.1

but need differentiability in (a,b)

(2) Thm 6.3.3 (& Thm 6.3.5) states only the case of taking limit

as • $x \to a^+$ (right hand limit) for "convenience".

In fact, it is true also for

- $x \to b^-$ (left hand limit)
- $x \to c$ (two-sided limit, $c \in (a,b)$)
- $x \to \pm\infty$

__Thm 6.3.3__ (__L'Hospital's Rule I__)

Let

- $-\infty \le a < b \le \infty$
- f, g differentiable on (a,b) (__NO__ assumption at end pts.)
- $g'(x) \neq 0, \ \forall x \in (a,b)$
- $\lim_{x \to a+} f(x) = 0 = \lim_{x \to a^+} g(x)$

(a) If $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, then $\lim_{x \to a+} \frac{f(x)}{g(x)} = L$

(b) If $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$, then $\lim_{x \to a+} \frac{f(x)}{g(x)} = L$

__Pf:__ For any α, β such that $a < \alpha < \beta < b$,

Rolle's implies $g(\beta) \neq g(\alpha)$ since $g'(x) \neq 0 \ \forall x \in (a,b)$.

Furthermore, Cauchy Mean Value Thm

$\Rightarrow \exists \ u \in (\alpha, \beta)$ such that

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(u)}{g'(u)} . \qquad \text{———} \ (*)$$

Case (a) $\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$

1-sided limit

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon, \ \forall x \in (a, a+\delta)$ $(a+\delta < b)$

If $a < \alpha < \beta < a+\delta$, then the u in $(*)$ satisfies

$$a < u < a+\delta.$$

Hence
$$L - \varepsilon < \frac{f'(u)}{g'(u)} < L + \varepsilon$$

$$\Rightarrow \quad L - \varepsilon < \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} < L + \varepsilon \qquad (\text{by } (*))$$

Letting $\alpha \to a+$ and using $\lim_{x \to a+} f(x) = 0 = \lim_{x \to a+} g(x)$,
we have $\forall \beta$ with $a < \beta < a+\delta$,

$$L - \varepsilon \le \frac{f(\beta)}{g(\beta)} \le L + \varepsilon$$

Now, $\forall \varepsilon' > 0$, we can choose $\varepsilon > 0$ s.t. $\varepsilon < \varepsilon'$.

Then $\left| \frac{f(\beta)}{g(\beta)} - L \right| \le \varepsilon < \varepsilon', \ \forall \beta \in (a, a+\delta)$.

In other words, $\forall \varepsilon' > 0$, $\exists \delta > 0$ s.t.

$$\left| \frac{f(\beta)}{g(\beta)} - L \right| < \varepsilon', \quad \forall \beta \in (a, a+\delta).$$

$$\therefore \quad \lim_{x \to a+} \frac{f(x)}{g(x)} = L.$$

Case (b) $\displaystyle \lim_{x \to a+} \frac{f'(x)}{g'(x)} = L$, $L = \pm\infty$.

If $L = +\infty$, then $\forall M > 0$, $\exists \delta > 0$ such that

$$\frac{f'(x)}{g'(x)} > M, \quad \forall x \in (a, a+\delta).$$

Hence for $a < \alpha < u < \beta < a+\delta$,

$$M < \frac{f'(u)}{g'(u)} = \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)}.$$

Letting $\alpha \to a^+$ & using $\displaystyle \lim_{x \to a+} f(x) = 0 = \lim_{x \to a+} g(x)$,

we have $\displaystyle M \leq \frac{f(\beta)}{g(\beta)}, \quad \forall a < \beta < a+\delta.$

Since $M > 0$ is arbitrary, we have $\displaystyle \lim_{x \to a+} \frac{f(x)}{g(x)} = +\infty = L$.

Similarly for $L = -\infty$ (check!) ※

Notes: (1) The cases of $\lim_{x\to\pm\infty}$ are in fact included in $x\to a^+$ & $x\to b^-$.

(2) The case of $\lim_{x\to b^-}$ can be proved similarly.

(3) Then, follow immediately, the case of $\lim_{x\to c}$.

(In this case, only need to assume $g'(x)\neq 0$ for $x\neq c$, $x\in(a,b)$)

see eg (b) in eg 6.3.4

eg 6.3.4

(a) $\lim_{x\to 0^+}\frac{\sin x}{\sqrt{x}}$ (note $\sqrt{x}$ is not differentiable at $x=0$)

$= \lim_{x\to 0^+}\frac{\cos x}{\frac{1}{2\sqrt{x}}}$ $\left(\begin{array}{l} f(x)=\sin x \text{ diff. \& } f'=\cos x \\ g(x)=\sqrt{x} \text{ diff (for } x>0) \\ \text{\& } g'(x)=\frac{1}{2\sqrt{x}}\neq 0,\ \forall x>0 \end{array}\right)$

$=0$ (limit exists, calculation justified)

(b) $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{\sin x}{2x}$?

indeterminate again

However, $f(x)=\sin x$ diff. & $f'(x)=\cos x$

$g(x)=2x$ diff. & $g'(x)=2\neq 0$ $\forall x\in\mathbb{R}$.

L'Hospital's Rule I (even the easier Thm 6.3.1) $\Rightarrow$

$$\lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2} \quad \text{has a limit.}$$

Hence L'Hospital's Rule I again $\Rightarrow$

$$\lim_{x\to 0+} \frac{1-\cos x}{x^2} = \lim_{x\to 0+} \frac{\sin x}{2x}$$

Since $(1-\cos x)' = \sin x$ exists & $(x^2)' = 2x \neq 0$, $\forall x > 0$

And $$\lim_{x\to 0-} \frac{1-\cos x}{x^2} = \lim_{x\to 0-} \frac{\sin x}{2x}$$

Since $\lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}$ exists, the 2 1-sided limits equal

and hence $$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}$$

(c) $$\lim_{x\to 0} \frac{e^x-1}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1. \quad \text{(check conditions!)}$$

As in (b), this existence of limit implies

$$\lim_{x\to 0} \left(\frac{e^x-1-x}{x^2}\right) = \lim_{x\to 0} \frac{e^x-1}{2x} = \underline{1} \quad \text{(check conditions!)}$$

(d) $\lim\limits_{x\to 1} \dfrac{\ln x}{x-1}$ (defined for $x>0$)

$= \lim\limits_{x\to 1} \dfrac{1/x}{1}$ $\left(\begin{array}{l} (\ln x)' = \frac{1}{x} \text{ exists } \forall x>0 \\ (x-1)' = 1 \text{ exists } \neq 0, \forall x>0 \end{array} \right)$

$= 1$ (limit exists, calculation justified)

<u>Thm 6.3.5</u> (<u>L'Hospital's Rule II</u>)

Let
- $-\infty \le a < b \le \infty$
- f, g differentiable on (a,b) (NO assumption at end pts.)
- $g'(x) \neq 0$, $\forall x \in (a,b)$
- $\lim\limits_{x\to a^+} g(x) = \pm\infty$

(a) If $\lim\limits_{x\to a+} \dfrac{f'(x)}{g'(x)} = L \in \mathbb{R}$, then $\lim\limits_{x\to a+} \dfrac{f(x)}{g(x)} = L$

(b) If $\lim\limits_{x\to a+} \dfrac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$, then $\lim\limits_{x\to a+} \dfrac{f(x)}{g(x)} = L$

Pf: Only for " $\lim\limits_{x\to a+} g(x) = +\infty$ ".

" $\lim\limits_{x\to a+} g(x) = -\infty$ " is similar.

As before, $\forall\ \alpha, \beta \in (a,b)$ with $a<\alpha<\beta<b$, we have

- $g(\beta) \neq g(\alpha)$ and
- $\dfrac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} = \dfrac{f'(u)}{g'(u)}$ for some $u \in (\alpha, \beta)$

Case (a): $L \in \mathbb{R}$.

subcase $L>0$

By $\lim\limits_{x\to a+} \dfrac{f'(x)}{g'(x)} = L$, $\forall\, \varepsilon>0$ $(\varepsilon < \frac{L}{2})$, $\exists\, \delta>0$ such that

$$0 < L-\varepsilon < \frac{f'(u)}{g'(u)} < L+\varepsilon, \quad \forall\ u \in (a, a+\delta) \quad (\& \ a+\delta<b)$$

$$\Rightarrow \quad L-\varepsilon < \frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} < L+\varepsilon, \quad \forall\ a<\alpha<\beta<a+\delta.$$

As $\lim\limits_{x\to a+} g(x) = +\infty$, $\exists\ c \in (a, a+\delta)$ such that

$$g(x) > 0, \quad \forall\ x \in (a,c) \quad (\subset (a, a+\delta))$$

Then for any $a<\alpha<c$, we have

$$L-\varepsilon < \frac{f(c)-f(\alpha)}{g(c)-g(\alpha)} < L+\varepsilon \quad (\text{by taking } \beta = c)$$

Using again $\lim\limits_{x\to a+} g(x) = +\infty$, we have

$$\lim_{\alpha\to a+} \frac{g(c)}{g(\alpha)} = 0$$

Therefore, $\exists\ 0 < c_1 < c$ such that

$$0 < \frac{g(c)}{g(\alpha)} < 1, \quad \forall\ \alpha \in (a, c_1)\ \ (\subset (a,c))$$

(Mistake in Textbook)

(Both $g(\alpha)$ & $g(c) > 0$ from above)

$$\therefore \quad \frac{g(\alpha) - g(c)}{g(\alpha)} = 1 - \frac{g(c)}{g(\alpha)} > 0, \quad \forall\ \alpha \in (a, c_1)$$

Therefore $$L - \varepsilon < \frac{f(c) - f(\alpha)}{g(c) - g(\alpha)} < L + \varepsilon,$$

$\Rightarrow$

$$(L-\varepsilon)\left(1 - \frac{g(c)}{g(\alpha)}\right) < \frac{f(c) - f(\alpha)}{g(c) - g(\alpha)} \cdot \left(1 - \frac{g(c)}{g(\alpha)}\right) < (L+\varepsilon)\left(1 - \frac{g(c)}{g(\alpha)}\right)$$

ie. $$(L-\varepsilon)\left(1 - \frac{g(c)}{g(\alpha)}\right) < \frac{f(\alpha)}{g(\alpha)} - \frac{f(c)}{g(\alpha)} < (L+\varepsilon)\left(1 - \frac{g(c)}{g(\alpha)}\right),$$

$\forall\ \alpha \in (a, c_1)$

which is

$$(L-\varepsilon)\left(1 - \frac{g(c)}{g(\alpha)}\right) + \frac{f(c)}{g(\alpha)} < \frac{f(\alpha)}{g(\alpha)} < (L+\varepsilon)\left(1 - \frac{g(c)}{g(\alpha)}\right) + \frac{f(c)}{g(\alpha)}, \quad \forall\ \alpha \in (a, c_1)$$

Using $\lim_{x \to a+} g(x) = +\infty$ again, $\exists\ c_2 \in (a, c_1)$ such that

$$0 < \frac{g(c)}{g(\alpha)} < \eta \quad \text{and} \quad 0 < \frac{|f(c)|}{g(\alpha)} < \eta, \quad \forall\ \alpha \in (a, c_2)$$

where $\eta = \min\{1, \varepsilon, \frac{\varepsilon}{L+1}\} > 0.$

Then $\frac{f(\alpha)}{g(\alpha)} < (L+\varepsilon)+\eta < L+2\varepsilon$

and $\frac{f(\alpha)}{g(\alpha)} > (L-\varepsilon)(1-\eta)-\eta$ (since $L+\varepsilon > L-\varepsilon > 0$)

$$= (L-\varepsilon) - [(L-\varepsilon)+1]\eta$$

$$\geq (L-\varepsilon) - (L+1-\varepsilon)\cdot\frac{\varepsilon}{L+1} \qquad \left(\eta \leq \frac{\varepsilon}{L+1}\right)$$

$$= L-\varepsilon-\varepsilon+\frac{\varepsilon^2}{L+1}$$

$$> L-2\varepsilon$$

We've proved that, $\forall\ 2\varepsilon > 0$ (equi. to $\forall \varepsilon > 0$) $(2\varepsilon < L)$

$\exists\ c_2 \in (a, c_1)$ such that

$$L-2\varepsilon < \frac{f(\alpha)}{g(\alpha)} < L+2\varepsilon, \quad \forall \alpha \in (a, c_2).$$

(c_2 can be written as $a+\delta$)

$$\therefore \lim_{x\to a^+}\frac{f(x)}{g(x)} = L.$$

The proof of the subcases that $L=0$ and $L<0$ are similar

(with careful consideration of "sign" in the inequalities!)

Or, by taking $\alpha \to a^+$ in (with $\lim_{\alpha\to a^+} g(\alpha) = +\infty$)

$$(L-\varepsilon)\left(1-\frac{g(c)}{g(\alpha)}\right)+\frac{f(c)}{g(\alpha)} < \frac{f(\alpha)}{g(\alpha)} < (L+\varepsilon)\left(1-\frac{g(c)}{g(\alpha)}\right)+\frac{f(c)}{g(\alpha)},$$

we have $$L-\varepsilon \leq \liminf_{\alpha\to a^+}\frac{f(\alpha)}{g(\alpha)} \leq \limsup_{\alpha\to a^+}\frac{f(\alpha)}{g(\alpha)} \leq L+\varepsilon$$

Since $\varepsilon>0$ $(\varepsilon<\frac{L}{2})$ is arbitrary, we have

$$L \leq \liminf_{\alpha\to a^+}\frac{f(\alpha)}{g(\alpha)} \leq \limsup_{\alpha\to a^+}\frac{f(\alpha)}{g(\alpha)} \leq L$$

$\Rightarrow$ $\lim\limits_{x\to a^+}\frac{f(x)}{g(x)}$ exists and equal L

(Pf of (b): next lecture)