eg 6.1.7 let $f: I \to \mathbb{R}$ is differentiable on I (ie at all points of I)

(a) Chain rule (also) $\Rightarrow$ $(f^n)'(x) = n\,(f(x))^{n-1} f'(x)$

(b) If further assume $f(x) \neq 0$, $\forall x \in I$, (mistake in textbook, $f' \neq 0$ not needed)

$$\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{(f(x))^2}, \quad \forall x \in I$$

by using $g(y) = \frac{1}{y}$ for $y \neq 0$ and $g'(y) = -\frac{1}{y^2}$, $\forall y \neq 0$.

(c) $$|f|'(x) = \operatorname{sgn}(f(x)) \cdot f'(x) = \begin{cases} f'(x), & \text{if } f(x) > 0 \\ -f'(x), & \text{if } f(x) < 0 \end{cases}$$

(where $\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ the signum function)

Pf: Consider $g(x) = |x|$. Then $g: (-\infty, \infty) \to \mathbb{R}$

and we've proved that g is differentiable at $x \neq 0$.

$$g'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

$\therefore$ For $x \neq 0$, $g'(x) = \operatorname{sgn}(x)$

(but $g' \neq \operatorname{sgn}$ at $x = 0$, because $g'(0)$ doesn't exist)

Therefore, by chain rule,

$|f|(x)$ is differentiable at x where $f(x) \neq 0$

and $\quad |f|'(x) = g'(f(x))\, f'(x) = \operatorname{sgn}(f(x))\, f'(x)$

$$= \begin{cases} f'(x), & f(x) > 0 \\ -f'(x), & f'(x) < 0 . \end{cases}$$

(At x where $f(x)=0$, the situation is more complicated:

(i) if $f(x)=x^2$, then $|f|(x) = x^2$ is differentiable also at $x=0$

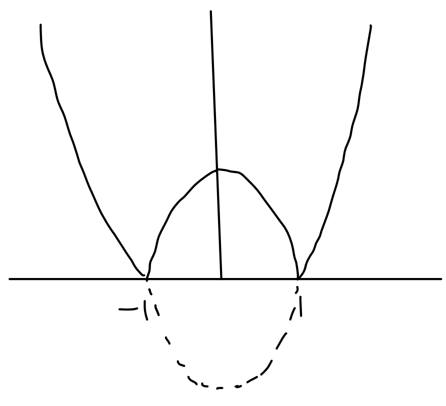
(ii) if $f(x)=x$, then $|f|(x) = |x|$ is not differentiable at $x=0$

See exercise 7 of §6.1 on page 171 of the textbook.)

Concrete example: $f(x) = x^2-1$, then $f(x)=0 \Leftrightarrow x=\pm 1$.

$\therefore \quad |f|(x) = |x^2-1|$ is differentiable for $x \neq \pm 1$ and

$$\frac{d}{dx}|x^2-1| = |f|'(x) = \operatorname{sgn}(x^2-1)\cdot 2x = \begin{cases} 2x, & \text{if } x<-1 \text{ or } x>1 \\ -2x, & \text{if } -1<x<1 \end{cases}$$



(d) <u>Derivatives of trigonometric functions</u>.

Let $S(x) = \sin x$, $C(x) = \cos x$ for $x \in \mathbb{R}$.

We'll define these two functions and prove the following later in section 8.4:

$$S'(x) = \cos x = C(x), \quad C'(x) = -\sin x = -S(x).$$

Using these facts & quotient rule, we have the formula for derivatives of other trigonometric functions:

$$\left.\begin{aligned} D \tan x &= (\sec x)^2 \\ D \sec x &= (\sec x)(\tan x) \end{aligned}\right\} \quad \text{for } x \neq \frac{(2k+1)\pi}{2}, \ k \in \mathbb{Z}$$

$$\left.\begin{aligned} D \cot x &= -(\csc x)^2 \\ D \csc x &= -(\csc x)(\cot x) \end{aligned}\right\} \quad \text{for } x \neq k\pi, \ k \in \mathbb{Z}$$

(e) $$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

By Chain rule, (product rule & quotient rule,) for $x \neq 0$

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \qquad (\text{check!})$$

But at $x = 0$, we must use definition of derivative to

find $$f'(0) = \lim_{\substack{x \to 0 \\ (x \neq 0)}} \frac{f(x) - f(0)}{x - 0} = \lim_{\substack{x \to 0 \\ (x \neq 0)}} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} = \lim_{\substack{x \to 0 \\ (x \neq 0)}} x \sin\frac{1}{x} = 0$$

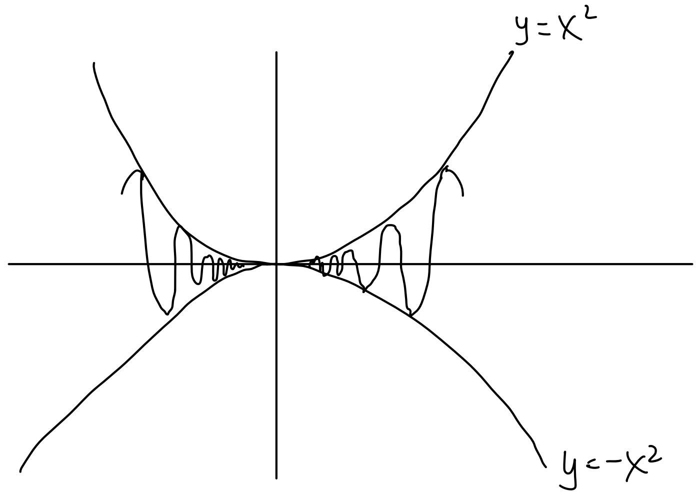
$\therefore$ $f'(x)$ exists for all $x \in \mathbb{R}$ and

$$f'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases}$$

(Note: However, $f'(x)$ is <u>discontinuous</u> at $x=0$

as $\lim\limits_{\substack{x\to 0 \\ (x\neq 0)}} \left(2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right)$ doesn't exist. (check)

$\therefore$ f differentiable $\forall x \not\Rightarrow f'$ is continuous.)



Inverse function

Thm 6.1.8 Let
- $I \subseteq \mathbb{R}$ be an interval
- $f: I \to \mathbb{R}$ be strictly monotone and continuous.
- $J = f(I)$ and $g: J \to \mathbb{R}$ be the strictly monotone & continuous function inverse to f.

If f is differentiable at $c \in I$ and $f'(c) \neq 0$, then g is differentiable at $d = f(c)$ and

$$g'(d) = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}$$

(Note $f'(c) \neq 0$ doesn't follow from f being strictly monotone:
eg. $f(x) = x^3$ is strictly monotone, but $f'(0) = 0$.
In this case, the inverse $g(x) = x^{\frac{1}{3}}$ is not differentiable at $x=0$.)

Pf: Since f is differentiable at $x=c$, Carathéodory's Thm 6.1.5
$\Rightarrow \exists\, \varphi: I \to \mathbb{R}$ with φ continuous at c such that

$$\begin{cases} f(x) - f(c) = \varphi(x)(x-c), \ \forall x \in I, \text{ and} \\ \varphi(c) = f'(c) \end{cases}$$

Since $f'(c) \neq 0$ and φ is continuous at c, $\exists\, \delta > 0$ such that

$$\varphi(x) \neq 0, \ \forall x \in (c-\delta, c+\delta) \cap I.$$

Let $U = f((c-\delta, c+\delta) \cap I) \subset J$

Then the inverse function g satisfies $f(g(y)) = y$, $\forall y \in U$.

Hence $y - d = f(g(y)) - f(c) = \varphi(g(y))(g(y) - c)$

$= \varphi(g(y))(g(y) - g(d))$ $\quad \left(\begin{array}{l} d = f(c) \in U \\ \Rightarrow c = g(d) \end{array} \right)$

Since $g(y) \in (c-\delta, c+\delta) \cap I$, $\forall y \in U$,

we have $\varphi(g(y)) \neq 0$.

Hence $g(y) - g(d) = \dfrac{1}{\varphi(g(y))}(y - d)$.

Since g is continuous on J and φ is continuous at $c = g(d)$ & $\neq 0$,

$1/\varphi \circ g$ is continuous at d.

Then by Carathéodory's Thm 6.1.5, g is differentiable at $d = f(c)$

and $g'(d) = \dfrac{1}{\varphi(g(d))} = \dfrac{1}{\varphi(c)} = \dfrac{1}{f'(c)}$. $\divideontimes$