Ch 6 Differentiation

§6.1 The Derivative

Def 6.1.1 • Let • $I \subseteq \mathbb{R}$ be an interval
• $f: I \to \mathbb{R}$ a function on I
• $c \in I$.

We say that $L \in \mathbb{R}$ is the derivative of f at c if $\forall \varepsilon > 0$, $\exists\ \delta(\varepsilon) > 0$ such that

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon, \quad \forall\ x \in I \text{ with } 0 < |x - c| < \delta(\varepsilon).$$

• In this case we say that f is differentiable at c, and we write $f'(c)$ for L.

Remark: • If limit exists, $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

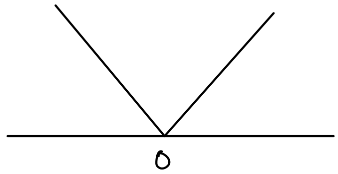
• c may be the endpoint of I (if I is "closed" at c)

then $\lim_{x \to c}$ means $\lim_{\substack{x \to c \\ x \in I}}$

• f' defines a function whose domain is a subset of I.

eg $f : (-\infty, \infty) \to \mathbb{R}$

$$\begin{array}{ccc} (-\infty,\infty) & \to & \mathbb{R} \\ \cup\!\!\!| & & \cup\!\!\!| \\ x & \longmapsto & f(x) = |x| \end{array}$$



Then $f' : (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$ given by

$$f'(x) = \begin{cases} 1 & , \ x \in (0, \infty) \\ -1 & , \ x \in (-\infty, 0) \end{cases}$$ and

$f'(0)$ doesn't exist (i.e. $|x|$ is not differentiable at $x = 0$)

Pf: For $c > 0$, then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{|x| - |c|}{x - c} = \lim_{x \to c} \frac{x - c}{x - c}$$

$$= \lim_{x \to c} 1 = 1 \qquad \left(\begin{array}{l} \text{as } x > 0 \\ \text{near } c > 0 \end{array}\right)$$

For $c < 0$, then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{|x| - |c|}{x - c} = \lim_{x \to c} \frac{-x + c}{x - c}$$

$$= \lim_{x \to c} -1 = -1 \qquad \left(\begin{array}{l} \text{as } x < 0 \\ \text{near } c < 0 \end{array}\right)$$

For $c = 0$, then

$$\lim_{\substack{x \to 0 \\ (x \neq 0)}} \frac{f(x) - f(c)}{x - c} = \lim_{x \to 0} \frac{|x|}{x} \quad \text{doesn't exist}$$

since the two one-sided limits are not equal:

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1 \neq 1 = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} \frac{|x|}{x}$$

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Note: The same argument shows that for $f(x)=x$, $x\in\mathbb{R}$,

f is differentiable $\forall x\in\mathbb{R}$ and

$$f'(x)=1, \quad \forall x\in\mathbb{R}.$$

> Thm 6.1.2 (Same notations as in Def 6.1.1)
>
> If $f: I\to\mathbb{R}$ has a derivative at $c\in I$ (i.e. differentiable at c),
>
> then f is continuous at c.

Pf: For $x\in I$ & $x\neq c$, we have

$$f(x)-f(c)=\frac{f(x)-f(c)}{x-c}\cdot(x-c)$$

$$f'(c) \text{ exists } \Rightarrow \lim_{\substack{x\to c\\(x\neq c)}}\left(f(x)-f(c)\right)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot\lim_{x\to c}(x-c)$$

$$=f'(c)\cdot 0=0$$

Hence $\lim_{x\to c} f(x)=f(c)$ $\therefore$ f is continuous at c. #

Remarks: • Previous eg $f(x)=|x|$ clearly shows that the converse of Thm 6.1.2 is not true (i.e. continuous at c $\not\Rightarrow$ differentiable at c)

• In fact, there exist continuous but nowhere differentiable functions (will be proved in MATH3060.)

Thm 6.1.3 (Same notations as in Def. 6.1.1)

Let $f: I \to \mathbb{R}$ & $g: I \to \mathbb{R}$ be functions that are differentiable at $c \in I$. Then

(a) If $\alpha \in \mathbb{R}$, the function αf is also differentiable at c, and

$$(\alpha f)'(c) = \alpha f'(c)$$

(b) The function $f+g$ is differentiable at c, and

$$(f+g)'(c) = f'(c) + g'(c)$$

(c) (Product Rule) The function fg is differentiable at c, and

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

(d) (Quotient Rule) If $g(c) \neq 0$, then the function f/g is differentiable at c, and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$$

(Pfs are easy, just using suitable algebraic expressions and taking limits, we just do the Quotient Rule here as example, you should do others by yourself.)

Pf of (d):

- Thm 6.1.2 implies that g is continuous at c (as g is diff. at c)
- Then $g(c) \neq 0 \Rightarrow$ there exists an interval $J \subseteq I$ with

  $c \in J$ such that $g(x) \neq 0, \forall x \in J$.

  (Thm 4.2.9 of the text book, MATH2050)

- $q = \frac{f}{g}$ is well-defined function on J and

  $\forall x \in J, \ x \neq c$, we have

$$\frac{q(x) - q(c)}{x - c} = \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c} = \frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)(x-c)}$$

$$= \frac{(f(x) - f(c))g(c) - f(c)(g(x) - g(c))}{g(x)g(c)(x-c)}$$

$$= \frac{1}{g(x)g(c)} \cdot \left[\frac{f(x) - f(c)}{x - c} \cdot g(c) - f(c) \cdot \frac{g(x) - g(c)}{x - c} \right]$$

f, g differentiable at $c \Rightarrow$ $\lim\limits_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$

$$\lim_{x \to c} \frac{g(x) - g(c)}{x - c} = g'(c)$$

$$\lim_{x \to c} g(x) = g(c) \ (\neq 0)$$

$\therefore$ $\lim\limits_{x \to c} \frac{q(x) - q(c)}{x - c}$ exists and

$$q'(c) = \frac{1}{g(c)^2} \left[f'(c)g(c) - f(c)g'(c) \right]$$

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$\boxed{\begin{array}{l}\underline{\text{Cor 6.1.4}}\ \text{If } f_1, \cdots, f_n \text{ are functions on an interval } I \text{ to } \mathbb{R} \\ \text{that are differentiable at } c \in I \text{, then} \\ \text{(a) The function } f_1 + \cdots + f_n \text{ is differentiable at } c\text{, and} \\ \qquad (f_1 + \cdots + f_n)'(c) = f_1'(c) + \cdots + f_n'(c) \\ \text{(b) The function } f_1 \cdots f_n \text{ is differentiable at } c\text{, and} \\ \qquad (f_1 \cdots f_n)'(c) = f_1'(c) f_2(c) \cdots f_n(c) + f_1(c) f_2'(c) \cdots f_n(c) \\ \qquad\qquad + \cdots + f_1(c) f_2(c) \cdots f_n'(c)\end{array}}$

Pf: Just by induction using Thm 6.1.3. ※

<u>Remark</u>: Quotient rule (Thm 6.1.3(d)) together with (b) in Cor 6.1.4

$$\Rightarrow \boxed{(x^n)' = n x^{n-1}, \ \forall n \in \mathbb{Z}} \quad (\forall x \neq 0 \text{ if } n < 0)$$

<u>Pf</u>: Applying (b) in Cor 6.1.4 to the case that

$$f_1 = \cdots = f_n = f \text{ (differentiable)},$$

$$\text{then}_{(\text{for } n \geq 1)} \ (f^n)' = (\underbrace{f \cdots f}_{n})' = \underbrace{f' \underbrace{f \cdots f}_{n-1} + f f' \cdots f + \cdots + \underbrace{f \cdots f}_{n-1} f'}_{n}$$

$$= n f^{n-1} f'.$$

We've proved that $(x)' = 1$, and hence

$$(x^n)' = n \cdot x^{n-1} \cdot 1 = n x^{n-1}$$

If $n=0$, then $f(x)=x^0 \equiv 1 \Rightarrow f'(c)=\lim\limits_{x\to c}\frac{f(x)-f(c)}{x-c}=0, \ \forall c$

$$\therefore (x^0)' \equiv 0 \equiv 0\cdot x^{-1}$$

(Note: strictly speaking, the RHS is not defined at $x=0$, but we may interpret the expression nx^{n-1} for $n=0$ as the continuous extension to the whole $\mathbb{R}$)

If $n=-m<0$ $(m>0)$, then for $x\neq 0$,

$$(x^n)' = \left(\frac{1}{x^m}\right)' = -\frac{(x^m)'}{(x^m)^2} \quad \text{by Quotient rule}$$

$$= -\frac{mx^{m-1}}{(x^m)^2} = (-m)\cdot x^{(-m)-1} = nx^{n-1} \quad (\text{for } x\neq 0) \quad \#$$

Chain Rule

Thm 6.1.5 (Carathéodory's Thm) (Same notations as in Def 6.1.1)

f is differentiable at c

$\Leftrightarrow \exists\ \varphi: I\to\mathbb{R}$ continuous at c such that

$$f(x)-f(c)=\varphi(x)(x-c), \ \forall x\in I.$$

In this case, $\varphi(c)=f'(c)$

Pf: $(\Rightarrow)$ If $f'(c)$ exists, define $\varphi: I\to\mathbb{R}$ by

$$\varphi(x)=\begin{cases}\dfrac{f(x)-f(c)}{x-c}, & x\neq c,\ x\in I\\[2mm] f'(c), & x=c.\end{cases}$$

Then $f'(c)$ exists $\Rightarrow$

$$\lim_{\substack{x \to c \\ (x \neq c)}} (\varphi(x) - \varphi(c)) = \lim_{x \to c} \left(\frac{f(x)-f(c)}{x-c} - f'(c) \right) = 0$$

$\therefore$ φ is continuous at c.

And clearly $f(x) - f(c) = \varphi(x)(x-c)$ for $x \neq c$,

which is also true trivially at $x=c$ since both sides equal zero.

($\Leftarrow$) If $\exists\, \varphi : I \to \mathbb{R}$ continuous at c such that

$$f(x) - f(c) = \varphi(x)(x-c), \quad \forall x \in I.$$

Then for $x \neq c$, $\dfrac{f(x)-f(c)}{x-c} = \varphi(x) \to \varphi(c)$ as $x \to c$

$\therefore$ $f'(c) = \lim_{x \to c} \dfrac{f(x)-f(c)}{x-c}$ exists & $= \varphi(c)$.

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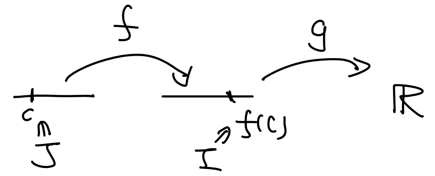
eg: $f(x) = x^3 : (-\infty, \infty) \to \mathbb{R}$

Then $f(x) - f(c) = x^3 - c^3 = (x^2 + cx + c^2)(x-c)$
$= \varphi(x)(x-c)$

where $\varphi(x) = x^2 + cx + c^2$ is continuous at c and

$\varphi(c) = 3c^2 = f'(c)$.

Thm 6.1.6 (Chain Rule)



Let
- I, J be intervals in $\mathbb{R}$,
- $g : I \to \mathbb{R}$
- $f : J \to \mathbb{R}$ with $f(J) \subseteq I$ (may just assume $f : J \to I$)
- $c \in J$.

If f is differentiable at c and g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and

$$(g\circ f)'(c) = g'(f(c))\, f'(c).$$

Other notations for f': Df or $\frac{df}{dx}$ (when x is the indep. variable)

The formula can be written as $(g\circ f)' = (g' \circ f)\cdot f'$ or

$$\boxed{D(g\circ f) = (Dg \circ f)\cdot Df}$$

Pf: Since $f'(c)$ exists, Carathéodory's Thm 6.1.5

$\Rightarrow \exists\ \varphi : J \to \mathbb{R}$ continuous at c such that

$$f(x) - f(c) = \varphi(x)(x-c), \quad \forall\ x \in J$$

and $\varphi(c) = f'(c)$.

Denote $f(c) = d$, then $g'(d)$ exists (similarly reasoning)

$\Rightarrow \exists\ \psi : I \to \mathbb{R}$ continuous at d such that

$$g(y)-g(d)=\psi(y)(y-d) \quad \forall\, y\in I$$

and $\psi(d)=g'(d)$.

For $x\in J$, substituting $y=f(x)$ & $d=f(c)$, we have

$$g(f(x))-g(f(c))=\psi(f(x))\,(f(x)-f(c))$$

$$\therefore \quad g\circ f(x)-g\circ f(c)=\psi(f(x))\,\varphi(x)(x-c)$$

$$=\left[(\psi\circ f)(x)\,\varphi(x)\right](x-c), \quad \forall\, x\in J$$

Since f diff. at c, f is continuous at c.

Together with ψ is continuous at $f(c)=d$, we have

$\psi\circ f$ is continuous at c.

Therefore $(\psi\circ f)(x)\,\varphi(x)$ is continuous at c (as φ is continuous at c)

$\therefore$ $g\circ f$ is differentiable at c by Carathéodory's Thm

and $(g\circ f)'(c)=(\psi\circ f)(c)\,\varphi(c)=\psi(d)\,f'(c)=g'(d)\,f'(c)$

$$=g'(f(c))\,f'(c).$$

Note: By using Carathéodory's Thm 6.1.5, we avoided the discussion of whether $f(x)-f(c)=0$ as in the usual proof by the algebraic expression

$$\frac{g(f(x))-g(f(c))}{x-c}=\frac{g(f(x))-g(f(c))}{f(x)-f(c)}\cdot\frac{f(x)-f(c)}{x-c}.$$