# MATH2060AB Homework 7
# Reference Solutions

7.4.12. Note that $f(x) = x^2$ is increasing on $[0, 1]$. Then, for the uniform partition $\mathcal{P}_n = (0, \frac{1}{n}, \frac{2}{n}, ..., 1 - \frac{1}{n}, 1)$, it holds that

$$L(f; \mathcal{P}_n) = (0^2 + 1^2 + \cdots + (n-1)^2)/n^3 = (n-1)n(2n-1)/6n^3 = \frac{1}{3}(1 - \frac{1}{n})(1 - \frac{1}{2n}),$$

and

$$U(f; \mathcal{P}_n) = (1^2 + 2^2 + \cdots + n^2)/n^3 = n(n+1)(2n+1)/6n^3 = \frac{1}{3}(1 + \frac{1}{n})(1 + \frac{1}{2n}).$$

Hence, $\lim_{n\to\infty} L(f; \mathcal{P}_n) = 1/3 = \lim_{n\to\infty} U(f; \mathcal{P}_n)$. Since $L(f; \mathcal{P}_n) \le L(f) \le U(f) \le U(f; \mathcal{P}_n)$ for any $n \ge 1$, it follows that $L(f) = U(f) = 1/3$.

7.4.13. Let $\mathcal{P}_\epsilon$ be the partition such that $U(f; \mathcal{P}_\epsilon) - L(f; \mathcal{P}_\epsilon) < \epsilon$ as stated in Integrability Criterion. It follows from Lemma 7.4.2 that if $\mathcal{P}$ is a refinement of $\mathcal{P}_\epsilon$, then $L(f; \mathcal{P}_\epsilon) \le L(f; \mathcal{P})$ and $U(f; \mathcal{P}) \le U(f; \mathcal{P}_\epsilon)$, so that $U(f; \mathcal{P}) - L(f; \mathcal{P}) \le U(f; \mathcal{P}_\epsilon) - L(f; \mathcal{P}_\epsilon) < \epsilon$.

8.1.5. Denote $f_n(x) = \sin(nx)/(1 + nx)$. If $x = 0$ then $f_n(0) = 0$ for all $n$ and hence $f_n(0) \to 0$ as $n \to \infty$. If $x > 0$, then $|f_n(x)| \le 1/(nx) \to 0$ and thus $f_n(x) \to 0$ as $n \to \infty$. Hence, $\lim_{n\to\infty} f_n(x) = 0$ for $x \ge 0$.

8.1.7. Denote $g_n(x) = e^{-nx}$. If $x = 0$ then $g_n(0) = 1$ for all $n$ and hence $g_n(0) \to 1$ as $n \to \infty$. If $x > 0$, by the fact that $0 < e^{-x} < 1$, $|g_n(x)| = (e^{-x})^n \to 0$ and hence $g_n(x) \to 0$ as $n \to \infty$.

8.1.15. Denote $f_n(x) = \sin(nx)/(1 + nx)$. For $\epsilon > 0$, let $N := [1/(a\epsilon)]+1$. Then for $n > N$ and $x \ge a$, we have $|f_n(x) - 0| \le 1/(an) < \epsilon$. Hence, $(f_n)$ converges uniformly on $[a, \infty)$ to 0.

Let $\epsilon_0 = (\sin 1)/2 > 0$, then for any $k \in N$, let $n_k = k$ and $x_k = 1/k \in [0, \infty)$, then $|f_{n_k}(x_k) - 0| = (\sin 1)/2 = \epsilon_0$. Therefore, $(f_n)$ does not converge uniformly on $[0, \infty)$ to 0.

8.1.17. Denote $g_n(x) = e^{-nx}$. For $\epsilon > 0$, let $N := [a^{-1} \ln \epsilon^{-1}] + 1$. Then for $n > N$ and $x \ge a$, we have $|g_n(x) - 0| = e^{-na} < \epsilon$. Hence, $(g_n)$ converges uniformly on $[a, \infty)$ to 0.

Let $\epsilon_0 = e^{-1} > 0$, then for any $k \in N$, let $n_k = k$ and $x_k = 1/k \in [0, \infty)$, then $|g_{n_k}(x_k) - 0| = e^{-1} = \epsilon_0$. Therefore, $(g_n)$ does not converge uniformly on $[0, \infty)$ to 0.

8.1.20. Note that there exists a constant $C > 0$ such that $0 \le x^3 e^{-x} \le C$ for any $x \ge 0$. For $x \ge a > 0$, we then have $n^2x^2e^{-nx} = \frac{1}{nx} \cdot (nx)^3 e^{-nx} \le \frac{C}{nx} \le \frac{C}{na}$. For $\epsilon > 0$, let $N := [C/(a\epsilon)] + 1$. Then for $n > N$ and $x \ge a$, we have $|n^2x^2e^{-nx} - 0| \le \frac{C}{na} < \epsilon$. Hence, $(n^2x^2e^{-nx})$ converges uniformly on $[a, \infty)$ to 0.

Let $\epsilon_0 = e^{-1} > 0$, then for any $k \in N$, let $n_k = k$ and $x_k = 1/k \in [0, \infty)$, then $|n_k^2 x_k^2 e^{-n_k x_k} - 0| = e^{-1} = \epsilon_0$. Therefore, $(n^2x^2e^{-nx})$ does not converge uniformly on $[0, \infty)$ to 0.