MATH2060AB Homework 5
Reference Solutions

7.2.3. Let $\dot{P}_n$ be the partition of $[0,1]$ into n equal subintervals with $t_1 = 1/n$ and $\dot{L}_n$ be the same subintervals tagged by irrational points. Then $S(H;\dot{P}_n) \geq 1 > 0 = S(H;\dot{L}_n)$. Hence, if H is Riemann integrable, then $\lim_{n\to\infty} S(H;\dot{P}_n) = \lim_{n\to\infty} S(H;\dot{L}_n)$, which is a contradiction.

7.2.5. Given any step function ϕ, we assume it takes distinct values $c_1, \cdots, c_n$, then $\phi^{-1}(c_j)$ is the union of a finite collection $\{J_{j1}, \cdots, J_{jr_j}\}$ by the definition of step function. Then one can write

$$\phi(x) = \sum_{j=1}^{n}\sum_{k=1}^{r_j} c_j \phi_{J_{jk}}(x).$$

7.2.6. ψ might not be a step function. Let $x \in [0,1]$, $\psi(x) = 1$ for all the rational numbers in $[0,1]$ and $\psi(x) = 0$ elsewhere. Then $\{x : \psi(x) = 1\}$ cannot be written as finite union of subintervals of $[0,1]$ since for any $[a,b] \in \{x : \psi(x) = 1\}$, then $a = b$, otherwise it contains irrational points which makes $\psi = 0$, but there are infinite rational numbers in $[0,1]$.

7.2.10. Let $h = f - g$, then h is continuous on $[a,b]$ and $\int_a^b h = 0$. If $h(a) = 0$, then $f(a) = g(a)$, otherwise, without loss of generality we may assume $h(a) > 0$. If there is no $c \in [a,b]$ such that $h(c) = 0$, then by intermediate value theorem we have $h(x) > 0$ for all $x \in [a,b]$, which yields $\int_a^b h > 0$ (See the first claim in 7.2.17). Then there is some $c \in [a,b]$ such that $h(c) = 0$, which gives $f(c) = g(c)$.

7.2.13. Let $f(x) := 1/x$ for $x \in (0,1]$ and $f(x) := 0$. Then $f \in \mathcal{R}[c,1]$ for every $c \in (0,1)$ but $f \notin \mathcal{R}[0,1]$ since f is not bounded.

7.2.15. Suppose $E = a = c_0 < c_1 < \cdots < c_m = b$, the case when $a \neq c_0$ or $b \neq c_m$ can be handled similarly. We first prove the restrictions of f to $[c_{i-1}, c_i]$ belong to $\mathcal{R}[c_{i-1}, c_i]$ for $i = 1, \cdots, m$. By the continuity property of f, it is Riemann integrable on $[c,d]$ for $c_{i-1} < c < d < c_i$. Let f be bounded by M on $c_{i-1} < c < d < c_i$, $\alpha_c(x) := -M$ and $\omega_c(x) := M$ for $x \in [c_{i-1}, c) \cup (d, c_i]$ and $\alpha_c(x) := \omega_c(x) := f(x)$ for $x \in (c,d)$. Then Theorem 7.2.9 implies $\alpha_c(x), \omega_c(x) \in \mathcal{R}[c_{i-1}, c_i]$. Moreover, $\int_{c_{i-1}}^{c_i} (\omega_c - \alpha_c) = 2M(c_i - d + c - c_{i-1}) < \epsilon$ when $\max\{c_i - d, c - c_{i-1}\} < \epsilon/4M$. Then Theorem 7.2.3 implies the restrictions of f to $[c_{i-1}, c_i]$ belong to $\mathcal{R}[c_{i-1}, c_i]$ for $i = 1, \cdots, m$. Hence, 7.2.14, which is a result by induction on Theorem 7.2.9 where the proof is omitted here, implies f is Riemann integrable on $[a,b]$.

7.2.17. We first claim $\int_a^b g(x)dx > 0$ since by continuity of g, there exists some $0 < \delta < (a+b)/4$ such that $g(x) > g((a+b)/2) > 0$ for $|x - (a+b)/2| \leq \delta$. Then $\int_a^b g(x)dx \geq \int_{(a+b)/2-\delta}^{(a+b)/2+\delta} g(x) \geq 2\delta g((a+b)/2) > 0$. Since f is continuous, there are constants m, M such that

$$m\int_a^b g \leq \int_a^b fg \leq M\int_a^b g.$$

Then by Theorem 5.3.7, there exists $c \in [a,b]$ such that

$$f(c) = \frac{\int_a^b fg}{\int_a^b g}.$$

If we do not have $g > 0$, without loss of generality we assume $[a,b] = [-1,1]$, let $g(x) = x$ and $f(x) = x$, we have $\int_{-1}^1 fg > 0 = f(c)\int_{-1}^1 g$, for any $c \in [-1,1]$.