MATH2060AB Homework 4
Reference Solutions

7.1.2. (a) The tag points are $(0, 1, 2)$ and

$$S(f, \dot{P}_1) = 9.$$

(b) The tag points are $(1, 2, 4)$ and

$$S(f, \dot{P}_1) = 37.$$

(c) The tag points are $(0, 2, 3)$ and

$$S(f, \dot{P}_2) = 13.$$

(d) The tag points are $(2, 3, 4)$ and

$$S(f, \dot{P}_2) = 33.$$

7.1.4. Let $\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ with $x_0 = 0$ and $x_n = 3$.

(a) $U_1 = \cup_{i=1}^m [x_{i-1}, x_i]$ for some m. For any $x \in U_1$, $x \in [x_{j-1}, x_j]$ for some $1 \le j \le m$. Then $x \ge x_{j-1} \ge x_0 = 0$ and $x \le x_j = t_j + x_j - t_j \le 1 + \|\dot{P}\|$ since $t_j \le 1$ and $|x_j - t_j| \le |x_j - x_{j-1}| \le \|\dot{P}\|$, which yields $U_1 \subseteq [0, 1 + \|\dot{P}\|]$.

For any $x \in [0, 1 - \|\dot{P}\|]$, $x \in [x_{j-1}, x_j]$ for some $j \ge 1$, then $t_j \ge x_{j-1} \ge 0$ and $t_j = x + t_j - x \le x + |x_j - x_{j-1}| \le 1 - \|\dot{P}\| + \|\dot{P}\| = 1$. Hence $x \in U_1$, which yields $[0, 1 - \|\dot{P}\|] \subseteq U_1$.

(b) $U_2 = \cup_{i=k}^l [x_{i-1}, x_i]$ for some k, l. For any $x \in U_2$, $x \in [x_{j-1}, x_j]$ for some $k \le j \le l$. Then $x = t_j + x - t_j \ge t_j - |x - t_j| \ge t_j - |x_j - x_{j-1}| \ge 1 - \|\dot{P}\|$ and $x \le x_j = t_j + x_j - t_j \le t_j + |x_j - x_{j-1}| \le 2 + \|\dot{P}\|$, which yields $U_1 \subseteq [1 - \|\dot{P}\|, 2 + \|\dot{P}\|]$.

For any $x \in [1 + \|\dot{P}\|, 2 - \|\dot{P}\|]$, $x \in [x_{j-1}, x_j]$ for some $j \ge k$, then $t_j \ge x_{j-1} + t_j - x_{j-1} \ge x_{j-1} - |x_j - x_{j-1}| \ge 1 + \|\dot{P}\| - \|\dot{P}\| = 1$ and $t_j = x + t_j - x \le x + |x_j - x_{j-1}| \le 2 - \|\dot{P}\| + \|\dot{P}\| = 1$. Hence $x \in U_2$, which yields $[1 + \|\dot{P}\|, 2 - \|\dot{P}\|] \subseteq U_2$.

7.1.7. When $n = 1$, the argument naturally holds. Suppose when $n = m$ it holds true, the for $n = m + 1$, by Theorem 7.1.5$(a), (b)$, $\sum_{i=1}^m k_i f_i + k_{m+1} f_{m+1} \in \mathcal{R}[a, b]$ and

$$\int_a^b f = \int_a^b \sum_{i=1}^{m+1} k_i f_i = \int_a^b (\sum_{i=1}^m k_i f_i + k_{m+1} f_{m+1}) = \int_a^b \sum_{i=1}^m k_i f_i + \int_a^b k_{m+1} f_{m+1} = \sum_{i=1}^m k_i \int_a^b f_i + k_{m+1} \int_a^b f_{k+1} = \sum_{i=1}^{k+1} k_i \int_a^b f_i.$$

7.1.9. For any $\epsilon > 0$, there exists some $\delta > 0$ such that if $\|\dot{P}\| \le \epsilon$, then

$$|S(f; \dot{P}) - \int_a^b f| < \epsilon.$$

By the fact that $\|\dot{P}_n\|$ tends to zero, then there exists some $N > 0$ such that for $n > N$,

$$|S(f; \dot{P}_n) - \int_a^b f| < \epsilon,$$

which gives

$$\int_a^b f = \lim_n S(f; \dot{P}_n).$$

7.1.11. If $f \in \mathcal{R}[a, b]$, then 7.1.9 implies $\int_a^b f = \lim_n S(f; \dot{P}_n) = \lim_n S(f; \dot{L}_n)$, which is a contradiction.

7.1.13. For any $\epsilon > 0$, let $\delta = \epsilon / 4\alpha$, then the union of the subintervals in $\|\dot{P}\|$ with tags in $[c, d]$ contains the interval $[c + \delta, d - \delta]$ and is contained in $[c - \delta, d + \delta]$, which gives

$$\alpha(d - c - 2\delta) \le S(\phi; \dot{P}) \le \alpha(d - c + 2\delta)$$

. Hence,

$$|S(\phi; \dot{P}) - \alpha(d - c)| \le 2\alpha\delta < \epsilon.$$

7.1.14. (a) Since $a \geq 0$, $x_j \geq 0$ for all $0 \leq j \leq n$, then

$$0 \leq x_{i-1} = \frac{1}{3}(x_{i-1}^2 + x_{i-1}x_{i-1} + x_{i-1}^2) \leq \frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2) \leq \frac{1}{3}(x_i^2 + x_i x_i + x_i^2) = x_i.$$

(b) Direct calculation gives $\frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2)(x_i - x_i - 1) = \frac{1}{3}(x_i^3 + x_i^2 x_{i-1} + x_i x_{i-1}^2 - x_i^2 x_{i-1} - x_i x_{i-1}^2 - x_{i-1}^3) = \frac{1}{3}(x_i^3 - x_{i-1}^3)$.
(c) By (b),

$$S(Q, \dot{L}) = \sum_{i=1}^{n} Q(q_i)(x_i - x_i - 1) = \sum_{i=1}^{n} \frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2)(x_i - x_i - 1) = \sum_{i=1}^{n} \frac{1}{3}(x_i^3 - x_{i-1}^3) = \frac{1}{3}(b^3 - a^3).$$

(d) The proof follows by the corresponding lines of Example 7.1.4 (c) after replacing $h(x), [0,1], q_i, \frac{1}{2}$ by $Q(x), [a,b], q_i = \frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2), \frac{1}{3}(b^3 - a^3)$ respectively.