MATH2060AB Homework 2
Reference Solutions

6.2.15. Since f' is bounded on I, then there exists some constant $C > 0$ such that

$$\sup_{x\in I} |f'(x)| < C.$$

Then for any $x, y \in I$, by the Mean Value Theorem, there exists some $c \in I$ such that

$$|f(x) - f(y)| = |f'(c)(x-y)| \le C|x-y|.$$

Hence, f satisfies a Lipschitz condition on I.

6.2.20. (a) By the Mean Value Theorem, there exists some $c_1 \in (0,1)$ such that

$$f'(c_1) = \frac{f(1)-f(0)}{1-0} = 1.$$

(b) By the Mean Value Theorem, there exists some $c_2 \in (1,2)$ such that

$$f'(c_2) = \frac{f(2)-f(1)}{2-1} = 0.$$

(c) Using the conclusions in (a) and (b), since $f'(c_2) < 1/3 < f'(c_1)$, by Darboux's Theorem, there exists some $c_3 \in (c_1, c_2) \subset (0,2)$ such that

$$f'(c_3) = 1/3.$$

6.3.3. For any $\epsilon > 0$, define $\delta := \epsilon$ such that for $|x - 0| < \epsilon$,

$$|\frac{f(x)-f(0)}{x-0}| \le |x| < \epsilon.$$

Then $f'(0) = 0$. Similarly we have $g'(0) = 0 = f'(0)$. Consider $f(x)/g(x) = \sin 1/x$. Define $x_n = 1/(2n\pi)$ and $y_n = 1/(\pi/2+2n\pi)$ for $n = 1, 2, \cdots$. Then $\lim_{n\to\infty} f(x_n)/g(x_n) = 0 \ne 1 = \lim_{n\to\infty} f(y_n)/g(y_n)$. Hence, $\lim_{x\to 0} f(x)/g(x)$ does not exist.

6.3.5. Since $\lim_{x\to 0}|x| = 0$ and $\lim_{x\to 0}|x/\sin x| = 1$, we have $\lim_{x\to 0}|x^2/\sin x| = 0$. By $0 \le |\frac{x^2 \sin\frac{1}{x}}{\sin x}| \le |\frac{x^2}{\sin x}|$ and Squeezing Theorem, it holds that

$$\lim_{x\to 0} |f(x)/g(x)| = \lim_{x\to 0} |\frac{x^2 \sin\frac{1}{x}}{\sin x}| = 0,$$

which yields $\lim_{x\to 0} f(x)/g(x) = 0$. Direct calculation gives that

$$f'(x)/g'(x) = \frac{2x\sin\frac{1}{x} - \cos\frac{1}{x}}{\cos x}.$$

Let $x_n = 1/(2n\pi)$ and $y_n = 1/(\pi/2 + 2n\pi)$, then

$$\lim_{n\to\infty} f'(x_n)/g'(x_n) = -1 \ne 0 = \lim_{n\to\infty} f'(y_n)/g'(y_n).$$

Hence, $\lim_{x\to 0} f(x)/g(x)$ does not exist.

6.3.6. (a) By $\lim_{x\to 0} e^x + e^{-x} - 2 = \lim_{x\to 0} 1 - \cos x = \lim_{x\to 0} e^x - e^{-x} = \lim_{x\to 0}\sin x = 0$, $\lim_{x\to 0} e^x + e^{-x} = 2$, $\lim_{x\to 0}\cos x = -1$, we use the L'Hospital's Rules to get

$$\lim_{x\to 0}\frac{e^x + e^{-x} - 2}{1-\cos x} = \lim_{x\to 0}\frac{e^x - e^{-x}}{\sin x} = \lim_{x\to 0}\frac{e^x + e^{-x}}{\cos x} = 2.$$

(b) Using similar arguments as in (a) and the L'Hospital's Rules, it holds that

$$\lim_{x\to 0}\frac{x^2 - \sin^2 x}{x^4} = \lim_{x\to 0}\frac{2x - 2\sin x\cos x}{4x^3} = \lim_{x\to 0}\frac{2 - 2\cos^2 x + 2\sin^2 x}{12x^2} = \lim_{x\to 0}\frac{\cos x \sin x}{3x} = \lim_{x\to 0}\frac{-\sin^2 x + \cos^2 x}{3} = \frac{1}{3}.$$

6.3.10. Consider $f(x) = \frac{e^x f(x)}{e^x}$. Then by the L'Hospital's Rules,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^x f(x) + e^x f'(x)}{e^x} = L.$$

Then

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} (f(x) + f'(x)) - \lim_{x\to\infty} f(x) = 0.$$

6.3.10. We use the L'Hospital's Rules to get

$$\begin{aligned}
&\lim_{x\to c} \frac{x^c - c^x}{x^x - c^c} \\
=& \lim_{x\to c} \frac{cx^{c-1} - c^x \ln c}{x^x(\ln x + 1)} \\
=& \frac{1 - \ln c}{\ln c + 1}.
\end{aligned}$$