MATH2060AB Homework 1
Reference Solutions

6.1.4. For any $\epsilon > 0$, we let $\delta = \epsilon$ such that for any rational $0 < |x - 0| < \delta$, it holds that

$$\left|\frac{f(x) - f(0)}{x - 0} - 0\right| = \left|\frac{x^2}{x} - 0\right| = |x| < \delta.$$

For any irrational $0 < |x - 0| < \delta$, it holds that

$$\left|\frac{f(x) - f(0)}{x - 0} - 0\right| = 0 < \delta.$$

Hence, f is differentiable at $x = 0$ and $f'(0) = 0$.

6.1.8. (a) f is differentiable for $x \in (-\infty, -1) \cup (-1, 0) \cup (0, \infty)$. For $x \in (-\infty, -1) \cup (-1, 0) \cup (0, \infty)$,

$$f'(x) = \begin{cases} -2 & \text{if } x \in (-\infty, -1) \\ 0 & \text{if } x \in (-1, 0) \\ 2 & \text{if } x \in (0, \infty). \end{cases}$$

(b) g is differentiable for $x \in (-\infty, 0) \cup (0, \infty)$. For $x \in (-\infty, 0) \cup (0, \infty)$,

$$g'(x) = \begin{cases} 1 & \text{if } x \in (-\infty, 0) \\ 3 & \text{if } x \in (0, \infty). \end{cases}$$

(c) h is differentiable for $x \in \mathbb{R}$.

$$h'(x) = \begin{cases} -2x & \text{if } x \in (-\infty, 0] \\ 2x & \text{if } x \in (0, \infty). \end{cases}$$

(d) k is differentiable for $x \in \cup_{n \in \mathbb{Z}} (n\pi, (n+1)\pi)$. For $x \in (n\pi, (n+1)\pi)$, $n \in \mathbb{Z}$

$$k'(x) = \begin{cases} \cos x & \text{if } n \text{ is even} \\ -\cos x & \text{if } n \text{ is odd}. \end{cases}$$

6.1.10. Given any $\epsilon > 0$, we let $\delta = \epsilon$l to get that for $0 < |x| < \delta$,

$$\left|\frac{g(x) - g(0)}{x - 0}\right| = \left|x \sin \frac{1}{x^2}\right| < |x| < \epsilon.$$

Hence, g is differentiable at $x = 0$ and $g'(0) = 0$. For any $x_0 \in (-\infty, 0) \cup (0, \infty)$, $\frac{1}{x^2}$ and x^2 are differentiable at $x = x_0$ and $\sin \frac{1}{x^2}$ is differentiable at $\frac{1}{x_0^2}$. Thus, $g(x)$ is differentiable for $x \in (-\infty, 0) \cup (0, \infty)$. Combining the above arguments, we proved that $g(x)$ is differentiable for $x \in \mathbb{R}$. Using the chain rule, we get for $x \neq 0$,

$$g'(x) = 2x \sin \frac{1}{x^2} - 2x^{-1} \cos \frac{1}{x^2}.$$

It is direct to verify that $g'(x_n)$ tends to $-\infty$ for

$$x_n := \sqrt{\frac{1}{2n\pi}}, \quad n \in \mathbb{Z}, \quad n \geq 1.$$

Hence $g'(x)$ is not bounded on $[-1, 1]$.

6.1.17. By definition, for given $\epsilon > 0$, there exists $\delta_0(\epsilon)$ such that if $x \in I$, then

$$|\frac{f(x)-f(c)}{x-c} - f'(c)| \leq \epsilon,$$

which yields

$$\left|f(u) - f(c) - f'(c)(u-c)\right| \leq \epsilon(c-u),$$

and

$$\left|f(v) - f(c) - f'(c)(v-c)\right| \leq \epsilon(v-c).$$

Then the triangle inequality gives that

$$\begin{aligned}&|f(v)-f(u)-(v-u)f'(c)| = |f(v)-f(c)-(v-c)f'(c)-f(u)+f(c)+(u-c)f'(c)|\\ \leq &|f(v)-f(c)-(v-c)f'(c)| + |f(u)-f(c)-(u-c)f'(c)| \leq \epsilon(v-u).\end{aligned}$$

Hence, it is direct to see $\delta(\epsilon) = \delta_0(\epsilon)$.

6.2.4. Since f is differentiable for $x \in \mathbb{R}$, the derivative of the point of relative minimum or maximum should be zero, then $f'(x_0) = \sum_{i=1}^n 2(x-a_i) - 0$ gives $x_0 = \frac{\sum_{i=1}^n a_i}{n}$, which is unique and a point of minimum since $f'(x) > 0$ for $x > x_0$ and $f'(x) < 0$ for $x < x_0$.

6.2.7. By the Mean Value Theorem, for $x > 1$, there exists a constant $1 \leq c \leq x$ such that

$$\ln x - \ln 1 = \frac{1}{c}(x-1),$$

which yields

$$\ln x \leq x - 1.$$

Similarly, for $x > 1$, there exists a constant $\frac{1}{x} \leq c \leq 1$ such that

$$\ln 1 - \ln\frac{1}{x} = -\frac{1}{c}(1-\frac{1}{x}) \geq 1 - \frac{1}{x},$$

which yields

$$\ln x \geq \frac{x-1}{x}.$$

6.2.9. $f(x) > 0$ for all $x \neq 0$, hence, f has an absolute minimum at $x = 0$. Direct calculation shows that

$$f'(x) = 8x^3 + 4x^3\sin\frac{1}{x} - x^2\cos\frac{1}{x}.$$

Let $n \in \mathbb{Z}$ and $n > 1$, it is also straightforward to verify that

$$f'(\frac{2}{\pi + 4n\pi}) > 0,$$

and

$$f'(-\frac{2}{3\pi+4n\pi}) < 0.$$

Any neighborhood of 0 contains $\frac{2}{\pi+4n_0\pi}$ and $-\frac{2}{3\pi+4n_0\pi}$ for some $n_0 \in \mathbb{Z}$ and $n_0 > 1$. Therefore, f' has both positive and negative values in every neighborhood of 0.