Topic #16 Series of Functions

Keywords

- Interchange of infinite summation with limit, integral, and differentiation
- Tests for Uniform convergence
  - Cauchy Criterion
  - Weierstrass M-Test
- Power series
  - radius of convergence
  - uniform convergence
  - integration
  - derivative
  - uniqueness
  - Taylor expansion

Def.

① An infinite series of functions $\sum f_n$ is **convergent on D** if the series $\sum_{n=1}^{\infty} f_n(x)$ is convergent, $\forall x \in D$,

namely,
$$\lim_{n\to\infty} \sum_{k=1}^{n} f_k(x) \text{ exists}, \quad \forall x \in D.$$

In such case, $\exists f: D \to \mathbb{R}$ s.t. $f(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x)$, $\forall x \in D$

We say: **$\sum f_n$ converges to f on D**, and write
$$f = \sum f_n \text{ on } D.$$

② $\sum f_n$ is **absolutely convergent on D** if $\sum_{n=1}^{\infty} |f_n(x)|$ is convergent for each $x \in D$.

③ $\sum f_n$ is **uniformly convergent on D** if $\left(\sum_{k=1}^{n} f_k\right)$ is uniformly convergent on D.

In such case, $\exists f: D \to \mathbb{R}$ s.t. $\sum_{k=1}^{n} f_k \rightrightarrows f$ on D

We say: $\sum f_n$ **converges to f uniformly on D**

Thm (Interchange of summation and limit)

If • $f_n : D \to \mathbb{R}$ continuous on D, $\forall n \in \mathbb{N}$

• $\sum_{k=1}^{n} f_k \rightrightarrows f$ on D

then f is continuous on D. Thus

$$\lim_{x \to x_0} \underbrace{\sum_{n=1}^{\infty} f_n(x)}_{f(x)} = \underbrace{\sum_{n=1}^{\infty} f_n(x_0)}_{f(x_0)}$$

Pf: Omitted.

Thm (Interchange of summation and integral)

If • $f_n \in \mathcal{R}[a,b]$, $\forall n \in \mathbb{N}$

• $\sum_{k=1}^{n} f_k \rightrightarrows f$ on $[a,b]$

then $f \in \mathcal{R}[a,b]$ with $\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$, i.e.,

$$\int_a^b \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_a^b f_n .$$

Pf: Omitted

Thm (Interchange of summation and derivative)

If • $\exists x_0 \in [a,b]$ s.t. $\sum_{n=1}^{\infty} f_n(x_0)$ is convergent

• $\sum_{k=1}^{n} f_k'$ is uniformly convergent on $[a,b]$

then $\exists f : [a,b] \to \mathbb{R}$ s.t.

• $\sum_{k=1}^{n} f_k \rightrightarrows f$ on $[a,b]$

• f is differentiable on $[a,b]$ with $f' = \sum f_n'$ on $[a,b]$

i.e.,

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} f_n(x)\right) = \sum_{n=1}^{\infty} \frac{d}{dx} f_n(x), \text{ on } [a,b]$$

Pf: Omitted.

Thm (Cauchy Criterion for uniform convergence of series of functions)

$\sum f_n$ is uniformly convergent on D $\iff$ $\forall \epsilon>0$, $\exists M(\epsilon)\in\mathbb{N}$ s.t.
if $m>n\geq M(\epsilon)$,
then $|f_{n+1}(x)+\cdots+f_m(x)|<\epsilon$, $\forall x\in D$

i.e. $\|f_{n+1}+\cdots+f_m\|_D<\epsilon$

Pf: direct. #

Thm (Weierstrass M-Test for uniform convergence)

If
- $\|f_n\|_D \leq M_n$, $\forall n\in\mathbb{N}$
- $\sum M_n$ is convergent

then $\sum f_n$ is uniformly convergent on D.

Pf: Omitted. Use Cauchy Criterion. #

Def (Power series)

$\sum_{n=0}^{\infty} a_n(x-c)^n$ is called a power series around $x=c$.

For simplicity, we treat only the case $c=0$.

Def. (Radius of Convergence, Interval of Convergence)

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series, then define

$$\rho = \begin{cases} \limsup_{n\to\infty} |a_n|^{1/n}, & \text{if } (|a_n|^{1/n}) \text{ is bounded;} \\ +\infty & \text{if } (|a_n|^{1/n}) \text{ is unbounded.} \end{cases}$$

and $$R \overset{\text{def.}}{=} \begin{cases} 0 & \text{if } \rho=+\infty \\ 1/\rho & \text{if } 0<\rho<+\infty \\ +\infty & \text{if } \rho=0 \end{cases}$$

We say: R is radius of convergence of $\sum a_n x^n$;
$(-R,R)$ is interval of convergence of $\sum a_n x^n$.

Notation: Let $(X_n)_{n\geq 0}$ be bounded.

Limit superior (or Limit supremum) of (X_n)

$$\limsup_{n\to\infty} x_n \overset{\text{def.}}{=} \lim_{n\to\infty}\left(\sup_{m\geq n} x_m\right)$$

$$= \inf_{n\geq 0} \sup_{m\geq n} x_m$$

(also write $\overline{\lim_{n\to\infty}}\, x_n$ = upper limit)

Fact: ① if $\limsup_{n\to\infty} x_n < x$ then $\exists N \in \mathbb{N}$ s.t. $x_n \leq x, \forall n \geq N$

② if $\limsup_{n\to\infty} X_n > X$ then $x_n \geq x$ for infinitely many $n \in \mathbb{N}$.

Thm (Cauchy-Hadamard Theorem)

Let $0 \leq R \leq +\infty$ be the radius of convergence of $\sum a_n x^n$, then

$\sum a_n x^n$ is absolutely convergent on $\{|x| < R\}$

$\sum a_n x^n$ is divergent on $\{|x| > R\}$

Pf: Treat only $0 < R < +\infty$ (Other cases: $R=0$ and $R=+\infty$ can be treated similarly)

① Let $0 < |x| < R$ (if $x=0$, then $\sum a_n x^n = 0$)

then $\exists\, c \in (0,1)$ s.t. $0 < |x| < cR$.

$\therefore \limsup_{n\to\infty} |a_n|^{1/n} = \rho = \frac{1}{R} < \frac{c}{|x|}$

then $\exists N \in \mathbb{N}$ s.t. $|a_n|^{\frac{1}{n}} \leq \frac{c}{|x|}$, $\forall n \geq N$

i.e. $|a_n x^n| \leq \left(\frac{c}{|x|}\right)^n \cdot |x|^n = c^n$

then $\sum |a_n x^n|$ is convergent (by Comparison Test)

$\therefore$ $\sum a_n x^n$ is absolutely convergent. #

② Let $|x| > R = \frac{1}{\rho}$, i.e. $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \rho > \frac{1}{|x|}$

$\therefore \exists \infty$-many n s.t. $|a_n|^{\frac{1}{n}} > \frac{1}{|x|}$, i.e. $|a_n x^n| > 1$

then $(a_n x^n)$ can NOT converge to zero

$\therefore \sum a_n x^n$ is divergent. ##

Remark (Important) It's convenient to use

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \quad \text{provided the limit exists}$$

(Exercise).

Thm (Uniform Convergence)

Let • R : the radius of convergence of $\sum a_n x^n$

• K : closed & bounded interval contained in $(-R, R)$

then

$\sum a_n x^n$ is uniformly convergent on K.

Pf: Note : $\exists\, 0 < c < 1$ s.t. $\sup_{x\in K} |x| < cR$

Similarly before, $\exists N \in \mathbb{N}$ s.t. $\sup_{x\in K} |a_n x^n| \leq c^n$, $\forall n \geq N$

Weierstrass M-test applies. #

Thm (Continuity, integrability, and differentiability)

Let R be the radius of convergence of $\sum a_n x^n$,

then

(a) $\sum a_n x^n$ is continuous on $(-R, R)$:

i.e. for each $x_0 \in (-R, R)$, $\lim_{x\to x_0} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x_0^n$.

bounded & closed interval

(b) $\sum a_n x^n \in \mathcal{R}[a,b]$ for any $[a,b] \subset (-R,R)$ with $-\infty<a<b<\infty$ and

$$\int_a^b \left(\sum_{n=0}^{\infty} a_n x^n\right) dx = \sum_{n=0}^{\infty} a_n \int_a^b x^n\, dx$$

(c) $\sum a_n x^n$ is differentiable on $(-R,R)$ with

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad |x|<R.$$

Pf: omitted.

Thm (Uniqueness)

If $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge to the same function f on $(-r,r)$ for some $r>0$, then

$$a_n = b_n, \quad \forall n \geq 0.$$

Pf: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $-R<x<R$, where $R>0$ is the radius of convergence.

Induction gives:

$f(x)$ is kth order differentiable ($k \in \mathbb{N}$) on $\{|x|<R\}$ with

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k}$$

note: $f^{(k)}$ is also continuous on $\{|x|<R\}$

$\therefore f^{(k)}(0) = k!\, a_k \quad (k \geq 0)$

Similarly, $f^{(k)}(0) = k!\, b_k \ (k \geq 0)$, then

$$a_k = b_k, \quad \forall k \geq 0. \quad \#$$

Def (Taylor Series)

Let f have derivatives of all orders at $c \in \mathbb{R}$, then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

is called the Taylor expansion of f at c.

Warning: Taylor expansion may NOT converge to the original function in an interval about c. $\boxed{\text{Ex. 9.4.12}}$

RK: Taylor's Theorem gives:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + R_n(x)$$

where $R_n(x) = \frac{f^{(n+1)}(x_*)}{(n+1)!}(x-c)^{n+1}$

for some x_* between x and c.

Thus, if $\exists R>0$ s.t. $R_n(x) \to 0$, $\forall x \in \{|x-c|<R\}$

then

$$\underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k}_{} \to f(x), \quad \forall x \in \{|x-c|<R\}$$

n^{th} Taylor polynomial for f at c

(it's also the partial sum of $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$)

In such case, we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n, \quad x \in \{|x-c|<R\}$$

for some $R>0$.

i.e. f can be represented as the Taylor expansion at c. #

Examples:

(a) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \forall x \in \mathbb{R}$

pf: Let $f(x) = \sin x$, $x \in \mathbb{R}$

then $\begin{cases} f^{(2n)}(x) = (-1)^n \sin x \\ f^{(2n+1)}(x) = (-1)^n \cos x \end{cases}$, $x \in \mathbb{R}$, $n \geq 0$

Therefore, at $c=0$,

$$f^{(2n)}(0) = 0, \quad f^{(2n+1)}(0) = (-1)^n$$

Taylor's Thm gives:

$$\sin x = \sum_{k=0}^{n} \frac{f^{(2k+1)}(0)}{(2k+1)!} x^{2k+1} + R_{2n+1}(x)$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1} + R_{2n+1}(x)$$

where $R_{2n+1}(x) = \frac{f^{(2n+2)}(x_*)}{(2n+2)!} x^{2n+2}$

Note: $\because f^{(2n+2)}(x_*) = (-1)^{n+1} \sin x_*$, $|\sin x_*| \le 1$

$\therefore |f^{(2n+2)}(x_*)| \le 1$

then

$$|R_{2n+1}(x)| \le \frac{x^{2n+2}}{(2n+2)!} \to 0, \ \forall x \in \mathbb{R}.$$

namely

$$\left| \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1} - \sin x \right| \to 0, \ \forall x \in \mathbb{R}$$

means: $\forall x \in \mathbb{R}$,

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sin x. \ \#$$

RK: Similarly, it holds:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \ \forall x \in \mathbb{R},$$

$$\left(\text{i.e.} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k}, \ \forall x \in \mathbb{R} \right)$$

(b) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\forall x \in \mathbb{R}$.

Pf: Let $g(x) = e^x$, $x \in \mathbb{R}$

then $g^{(n)}(x) = e^x$, $x \in \mathbb{R}$, $n \ge 0$.

Taylor's Thm gives: $\forall x \in \mathbb{R}$,

$$e^x = g(x) = \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} x^k + R_n(x)$$

$$= \sum_{k=0}^{n} \frac{x^k}{k!} + R_n(x)$$

where $R_n(x) = \frac{g^{(n+1)}(x_*)}{(n+1)!} x^{n+1}$

for some x_* between 0 and x

Note: $|R_n(x)| = \left| \frac{e^{x_*}}{(n+1)!} x^{n+1} \right| \le e^{|x|} \frac{|x|^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$

This shows: e^x can be given as its Taylor expansion at $c=0$:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad \forall x \in \mathbb{R}. \quad \#$$

RK: For $c \in \mathbb{R}$,

$$e^x = e^c \cdot e^{x-c} = e^c \sum_{n=0}^{\infty} \frac{1}{n!} (x-c)^n$$

$$= \sum_{n=0}^{\infty} \frac{e^c}{n!} (x-c)^n, \quad \forall x \in \mathbb{R}. \quad \#$$