Topic #15 Tests for Non-absolute Convergence

Keywords:
- Alternating Series Test
- Dirichlet's Test
- Abel's Test

Def. (Alternating Series)
$\sum x_n$ is an alternating series if the terms $(-1)^{n+1} x_n$, $n \in \mathbb{N}$, are all positive (or all negative).

Thm (Alternating Series Test)
An alternating series $\sum (-1)^{n+1} z_n$ with $\begin{cases} \cdot\ z_n > 0,\ \forall n \in \mathbb{N} \\ \cdot\ (z_n) \text{ is decreasing such that } \lim\limits_{n\to\infty} z_n = 0 \end{cases}$

is convergent

Pf: Consider the partial sum S_{2n}:

$$S_{2n} = (z_1 - z_2) + (z_3 - z_4) + \cdots + (z_{2n-1} - z_{2n})$$

Note: ① $\because$ (z_n) is decreasing

$\therefore$ $z_k - z_{k+1} \geq 0$, $\forall k \in \mathbb{N}$

then (S_{2n}) is increasing

② Rewrite S_{2n} as

$$S_{2n} = z_1 - (z_2 - z_3) - \cdots - (z_{2n-2} - z_{2n-1}) - z_{2n}$$

$\therefore$ $S_{2n} \leq z_1$, $\forall n \in \mathbb{N}$

Monotone Convergence Thm gives:

$$\exists S \in \mathbb{R} \text{ s.t. } S = \lim_{n\to\infty} S_{2n}$$

To show $\lim\limits_{n\to\infty} S_n = S$, it suffices to show:

$$\lim_{n\to\infty} S_{2n+1} = S$$

In fact,

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (S_{2n} + z_{2n+1})$$
$$= \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} z_{2n+1} = S + 0 = S. \quad \#$$

Example:

Recall: p-series $\sum \frac{1}{n^p}$ $\begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \le 1 \end{cases}$

By Alternating Series Test

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} \text{ is convergent if } p > 0.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} \text{ is } \begin{cases} \text{conditionally convergent if } 0 < p \le 1 \\ \text{absolutely convergent if } p > 1 \\ \text{divergent if } p \le 0. \end{cases}$$

Thm (Dirichlet's Test)

$\sum x_n y_n$ with
- (x_n) is decreasing such that $\lim_{n\to\infty} x_n = 0$
- partial sums (S_n) of $\sum y_n$ are bounded,
  i.e., $\exists B \in \mathbb{R}$ s.t. $|S_n| \le B, \forall n \in \mathbb{N}$

is convergent

Pf. claim (Abel's lemma) (or partial summation formula):

$\forall m > n$, $$\sum_{k=n+1}^{m} x_k y_k = (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k.$$

In fact,
$$\sum_{k=n+1}^{m} x_k y_k = \sum_{k=n+1}^{m} x_k (S_k - S_{k-1})$$
$$= \sum_{k=n+1}^{m} x_k S_k - \sum_{k=n+1}^{m} x_k S_{k-1}$$
$$= \left(\sum_{k=n+1}^{m-1} x_k S_k + x_m S_m \right) - \left(x_{n+1} S_n + \sum_{k=n+2}^{m} x_k S_{k-1} \right)$$
$$= (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k$$

(note: $n+2 \le k \le m \Rightarrow n+1 \le k-1 \overset{\text{def}}{=} k' \le m-1$) #

Therefore,

$$\left|\sum_{k=n+1}^{m} x_k y_k\right| \leq (x_m + x_{n+1}) B + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) B$$

(note: $x_k - x_{k+1} \geq 0$)

$$= [(x_m + x_{n+1}) + (x_{n+1} - x_m)] B$$

$$= 2 x_{n+1} B \to 0 \text{ as } n \to \infty$$

Cauchy Convergence Criterion implies:

$\sum x_k y_k$ is convergent. ##

Thm (Abel's Test)

$\sum x_n y_n$ is convergent if

- (x_n) is monotone and convergent
- $\sum y_n$ is convergent.

Pf: Case: (x_n) is decreasing with $x = \lim_{n\to\infty} x_n$

Define $u_n = x_n - x$

then (u_n) is decreasing with $\lim_{n\to\infty} u_n = 0$

Write:

$$\sum x_n y_n = \sum (x + u_n) y_n = x \sum y_n + \sum u_n y_n$$

where
- $\sum y_n$ is convergent
- $\sum u_n y_n$ is convergent by Dirichlet's Test.

(note: $\because$ $\sum y_n$ is convergent

$\therefore$ partial sums of $\sum y_n$ are bounded)

Therefore, $\sum x_n y_n$ is convergent

Case: (x_n) is increasing with $x = \lim_{n\to\infty} x_n$

Define $u_n = x - x_n$, so $\sum x_n y_n = \sum (x - u_n) y_n$

$$= x \sum y_n - \sum u_n y_n.$$

The rest is similar. ##

Examples (Apply Dirichlet's Test)

Assume: (a_n) is decreasing with $\lim_{n\to\infty} a_n = 0$

(a) $\sum_{n=1}^{\infty} a_n \cos(nx)$ is convergent if $x \neq 2k\pi$ $(k \in \mathbb{N})$.

Pf: Note:

$$2(\sin\tfrac{1}{2}x)(\cos x + \cdots + \cos nx) = \sin(n+\tfrac{1}{2})x - \sin\tfrac{1}{2}x$$

(Exercise)

Then, for $x \neq 2k\pi$ $(k \in \mathbb{N})$,

$$|\cos x + \cdots + \cos nx| = \left| \frac{\sin(n+\frac{1}{2})x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x} \right|$$

$$\leq \frac{1}{|\sin\frac{1}{2}x|}$$

$\therefore \left(\sum_{k=1}^{n} \cos kx\right)$ is bounded.

Dirichlet's Test gives:

$\sum_{n=1}^{\infty} a_n \cos nx$ is convergent. #

(b) $\sum_{n=1}^{\infty} a_n \sin nx$ is also convergent if $x \neq 2k\pi$ $(k \in \mathbb{N})$

(Of course, it is trivial to see: if $x = 2k\pi$ with $k \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n \sin nx = 0$ is also convergent)

Pf: Note (again exercise):

$$2(\sin\tfrac{1}{2}x)(\sin x + \cdots + \sin nx) = \cos\tfrac{1}{2}x - \cos(n+\tfrac{1}{2})x$$

Then, for $x \neq 2k\pi$ $(k \in \mathbb{N})$,

$$|\sin x + \cdots + \sin nx| \leq \frac{1}{|\sin\frac{1}{2}x|}$$

Dirichlet's Test applies. #