Topic #14 Tests for Absolute Convergence

Keyword: – Limit Comparison Test: $\lim \left|\frac{x_n}{y_n}\right|$
– Root Test: $|x_n|^{\frac{1}{n}}$, $\lim_n |x_n|^{\frac{1}{n}}$
– Ratio Test: $\left|\frac{x_{n+1}}{x_n}\right|$, $\lim \left|\frac{x_{n+1}}{x_n}\right|$
– Integral Test: $\sum x_n$, $x_n = f(n)$, $\int_1^\infty f(t)\,dt$
– Raabe's Test: $n\left(1 - \left|\frac{x_{n+1}}{x_n}\right|\right)$, $\lim n\left(1 - \left|\frac{x_{n+1}}{x_n}\right|\right)$

Thm (Limit Comparison Test, II)
Assume • $x_n \neq 0$, $y_n \neq 0$
• $r \overset{\text{def.}}{=} \lim_{n\to\infty} \left|\frac{x_n}{y_n}\right|$ exists (≥ 0)
If $r \neq 0$ $(\because r > 0)$, then
$\sum x_n$ is absolutely convergent $\Leftrightarrow$ $\sum y_n$ is absolutely convergent
If $r = 0$, then
$\sum y_n$ is absolutely convergent $\Rightarrow$ $\sum x_n$ is absolutely convergent.

Pf: omitted.

e.g. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ is convergent $\left(\because S_n = 1 - \frac{1}{n+1}\right)$ $\to 1$ as $n\to\infty$

$$\left|\frac{\frac{1}{n^2}}{\frac{1}{n(n+1)}}\right| = \frac{1}{n^2}\cdot n(n+1) = \frac{n+1}{n} = 1 + \frac{1}{n} \to 1 \neq 0$$

$\therefore \sum \frac{1}{n^2}$ is convergent.

Thm (Root Test) (Cauchy)
(a) • If $\exists\, r < 1$ and $K \in \mathbb{N}$ s.t. $|x_n|^{\frac{1}{n}} \leq r$, $\forall n \geq K$,
then $\sum x_n$ is absolutely convergent.

(b) • If $\exists K \in \mathbb{N}$ s.t. $|x_n|^{\frac{1}{n}} \geq 1$, $\forall n \geq K$,
then $\sum x_n$ is divergent

pf: (a) $\because |x_n|^{\frac{1}{n}} \le r, \forall n \ge K$

$\therefore |x_n| \le r^n, \forall n \ge K$

note: $\sum r^n$ is convergent for $0 \le r < 1$

then Comparison Test (3.7.7 of textbook) tells:

$\sum |x_n|$ is convergent.

(b) $\because |x_n|^{\frac{1}{n}} \ge 1, \forall n \ge K$

$\therefore |x_n| \ge 1, \forall n \ge K$

then it's NOT true to have: $\lim x_n = 0$

The nth Term Test tells: $\sum x_n$ is divergent. #

Coro (limit version of Root Test)

Assume: $r \overset{\text{def.}}{=} \lim |x_n|^{\frac{1}{n}}$ exists

then if $r < 1$, then $\sum x_n$ is absolutely convergent;

if $r > 1$, then $\sum x_n$ is divergent.

pf: • Let $r < 1$, then

$\exists r_1$ with $r < r_1 < 1$ and $K \in \mathbb{N}$ s.t. $|x_n|^{\frac{1}{n}} < r_1, \forall n \ge K$.

• Let $r > 1$, then

$\exists K \in \mathbb{N}$ s.t. $|x_n|^{\frac{1}{n}} > 1, \forall n \ge K$. #

Note: It says nothing for $r = 1$.

If $r = 1$, either convergence or non-convergence is possible.

Thm (Ratio Test) (D'Alembert)

Let $x_n \ne 0, \forall n \in \mathbb{N}$.

(a) If $\exists\, 0 < r < 1$ and $K \in \mathbb{N}$ s.t. $\left|\frac{x_{n+1}}{x_n}\right| \le r, \forall n \ge K$,

then $\sum x_n$ is absolutely convergent

(b) If $\exists K \in \mathbb{N}$ s.t. $\left|\frac{x_{n+1}}{x_n}\right| \ge 1, \forall n \ge K$,

then $\sum x_n$ is divergent.

Pf: (a) Let $\left|\frac{x_{n+1}}{x_n}\right| \le r, \ \forall\, n \ge K$

then Induction gives: $|x_{K+m}| \le |x_K|\, r^m, \ \forall\, m \in \mathbb{N}$

note: $\sum r^m$ is convergent for $0<r<1$

then $\sum_m |x_{K+m}|$ is convergent

Comparison Test tells: $\sum_n |x_n|$ is convergent

(b) Let $\left|\frac{x_{n+1}}{x_n}\right| \ge 1, \ \forall\, n \ge K$

then Induction gives: $|x_{K+m}| \ge |x_K|, \ \forall\, m \in \mathbb{N}$

note: $|x_K| > 0$

then n^{th} Term Test applies. #

Coro (limit version of Ratio Test)

Let • $x_n \neq 0, \ \forall\, n \in \mathbb{N}$

• $r \overset{\text{def.}}{=} \lim \left|\frac{x_{n+1}}{x_n}\right|$ exists

then (a) if $r<1$, then $\sum x_n$ is absolutely convergent,

(b) if $r>1$, then $\sum x_n$ is divergent

Pf: omitted.

note: Again, if $r=1$, then either conv. or non-conv. is possible.

Thm (Integral Test)

Let $f(t)$ be positive and decreasing on $\{t : t \ge 1\}$

then $\sum_{k=1}^{\infty} f(k)$ is convergent $\iff$ $\int_1^{\infty} f(t)\,dt \overset{\text{def.}}{=} \lim_{b\to\infty} \int_1^b f(t)\,dt$ exists

In such case,

$$\int_{n+1}^{\infty} f(t)\,dt \le \sum_{k=1}^{\infty} f(k) - \sum_{k=1}^{n} f(k) \le \int_n^{\infty} f(t)\,dt, \quad \forall\, n \in \mathbb{N}.$$

note: $\sum_{k=1}^{n} f(k) \to \sum_{k=1}^{\infty} f(k)$ as $n \to \infty$

Pf: $\because f$ is decreasing on $[1, \infty)$

$\therefore f(k) \le f(t) \le f(k-1), \ \forall t \in [k-1, k], \ k = 2, 3, \cdots$

then

$$\int_{k-1}^{k} f(k)\, dt \le \int_{k-1}^{k} f(t)\, dt \le \int_{k-1}^{k} f(k-1)\, dt$$

(*) i.e. $f(k) \le \int_{k-1}^{k} f(t)\, dt \le f(k-1), \quad k = 2, 3, \cdots$

Take $\sum_{k=2}^{n}$ gives:

$$\sum_{k=2}^{n} f(k) \le \sum_{k=2}^{n} \int_{k-1}^{k} f(t)\, dt \le \sum_{k=2}^{n} f(k-1)$$

implies:

$$S_n - f(1) \le \int_1^n f(t)\, dt \le S_{n-1}$$

where $S_n = \sum_{k=1}^{n} f(k)$ is the partial sum.

note: $\because f > 0$ on $[1, \infty)$

$\therefore S_n > 0$ increasing; $\int_1^n f(t) dt > 0$ increasing

Therefore

$$\lim_{n\to\infty} S_n \text{ exists} \iff \lim_{n\to\infty} \int_1^n f(t)\, dt \text{ exists}$$

i.e. $\sum_{k=1}^{\infty} f(k)$ is convergent $\iff \int_1^{\infty} f(t)\, dt$ exists

(Exercise: $\lim_{n\to\infty} \int_1^n f(t) dt$ exists $\iff \lim_{b\to\infty} \int_1^b f(t) dt$ exists)

In case of convergence, taking $\sum_{k=n+1}^{m}$ $(m > n)$ in (*) gives:

$$S_m - S_n \le \int_n^m f(t)\, dt \le S_{m-1} - S_{n-1}$$

Rewrite it as

$$\int_{n+1}^{m+1} f(t) dt \le S_m - S_n \le \int_n^m f(t)\, dt, \quad \forall m > n$$

Let $m \to \infty$, then $\int_{n+1}^{\infty} f(t) dt \le S - S_n \le \int_n^{\infty} f(t)\, dt, \ \forall n \ge 1$

where $S = \lim_{m\to\infty} S_m = \sum_{k=1}^{\infty} f(k)$.

#

e.g. p-series: $\sum \frac{1}{n^p} = \sum_{k=1}^{\infty} f(k)$ with $f(t) \overset{\text{def.}}{=} \frac{1}{t^p}$, $t \geq 1$.

<u>note</u>: $p=1$: $\int_1^n f(t)\,dt = \int_1^n \frac{1}{t}\,dt = \ln n - \ln 1$

$p \neq 1$: $\int_1^n f(t)\,dt = \int_1^n \frac{1}{t^p}\,dt = \frac{1}{1-p}\left(\frac{1}{n^{p-1}} - 1\right)$

<u>Thus</u>: $\sum \frac{1}{n^p}$ is convergent

$\iff \lim_{n\to\infty} \int_1^n f(t)\,dt$ exists

$\iff p-1>0$, i.e. $p>1$

Then $\sum \frac{1}{n^p}$ converges if $p>1$;

diverges if $p \leq 1$. #

<u>RK</u>: Root Test or Ratio Test fails for the p-series. #

<u>Thm</u> (<u>Raabe's Test</u>)

Let $x_n \neq 0$, $\forall n \in \mathbb{N}$, then

(a) If $\exists\, a>1$ and $K \in \mathbb{N}$ s.t. $\left|\frac{x_{n+1}}{x_n}\right| \leq 1 - \frac{a}{n}$, $\forall n \geq K$,

then $\sum x_n$ is absolutely convergent;

(b) If $\exists\, a \leq 1$ and $K \in \mathbb{N}$ s.t. $\left|\frac{x_{n+1}}{x_n}\right| \geq 1 - \frac{a}{n}$, $\forall n \geq K$

then $\sum x_n$ is NOT absolutely convergent,

(i.e. $\sum |x_n|$ diverges).

<u>pf</u>: Omitted.

<u>Coro</u> (limit form of Raabe's Test)

Let $x_n \neq 0$, $\forall n \in \mathbb{N}$ and $a \overset{\text{def.}}{=} \lim_{n\to\infty} n\left(1 - \left|\frac{x_{n+1}}{x_n}\right|\right)$ exists,

then (a) if $a>1$ then $\sum x_n$ is absolutely convergent;

(b) if $a<1$ then $\sum x_n$ is NOT absolutely convergent.

NOTE: Corollary tells nothing for $a=1$.

Exampls:

(a) Raabe's Test for $\sum \frac{1}{n^p}$:

$$a = \lim_{n\to\infty} n\left(1 - \left|\frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}}\right|\right) = \lim_{n\to\infty} n\left[1 - \frac{n^p}{(n+1)^p}\right]$$

$$= \lim_{n\to\infty} n\left[1 - \frac{1}{(1+\frac{1}{n})^p}\right] = \lim_{n\to\infty}\left[\frac{(1+\frac{1}{n})^p - 1}{\frac{1}{n}} \cdot \frac{1}{(1+\frac{1}{n})^p}\right]$$

note: $\displaystyle \lim_{n\to\infty} \frac{(1+\frac{1}{n})^p - 1}{\frac{1}{n}} = \frac{d}{dx}\Big|_{x=1} x^p = p$

$\therefore\ a = 1 \cdot p = p$

therefore

$p>1 \Rightarrow \sum \frac{1}{n^p}$ is absolutely convergent

$p<1 \Rightarrow \sum \frac{1}{n^p}$ is NOT absolutely convergent

(~~then~~ NOT convergent)

However, Raabe's Test (limiting form) fails for $p=1$.

(b) $\displaystyle \sum_{n=1}^{\infty} \frac{n}{n^2+1}$

note:

- $\displaystyle \left|\frac{x_{n+1}}{x_n}\right| = \frac{n+1}{(n+1)^2+1} \cdot \frac{n^2+1}{n} = \frac{n+1}{n} \cdot \frac{n^2+1}{(n+1)^2+1} \to 1$

- $\displaystyle n\left(1 - \left|\frac{x_{n+1}}{x_n}\right|\right) = n \cdot \left[1 - \frac{n+1}{n} \cdot \frac{n^2+1}{(n+1)^2+1}\right]$

$$= \frac{n^2+n-1}{(n+1)^2+1} \to 1$$

then both Ratio Test (limiting form)

and Raabe's Test (limiting form)

fails.

But: $\displaystyle \left|\frac{x_{n+1}}{x_n}\right| - 1 = \frac{n+1}{n}\,\frac{n^2+1}{(n+1)^2+1} - 1 = \frac{(n+1)(n^2+1) - n[(n+1)^2+1]}{n[(n+1)^2+1]}$

$$= -\frac{n^2+n-1}{n[(n+1)^2+1]} = -\frac{1}{n} \cdot \frac{n^2+n-1}{n^2+2n+2} \geq -\frac{1}{n}$$

i.e. $\left|\frac{x_{n+1}}{x_n}\right| \geqslant 1 - \frac{1}{n}, \ \forall n \geq 1.$ $\quad \begin{pmatrix} a = 1 \leq 1 \\ K = 1 \in \mathbb{N} \end{pmatrix}$

then Raabe's Test applies: $\sum x_n$ is not absolutely convergent

($\because x_n > 0$

$\therefore \sum x_n$ is not convergent)

RK: Limit Comparison Test with $y_n = \frac{1}{n}$ also applies. #