

Topic #13 Absolute Convergence

Def.: Let (x_n) be a sequence in \mathbb{R} ,

- $\sum x_n$ is **absolutely convergent** if $\sum |x_n|$ is convergent in \mathbb{R} ;
- $\sum x_n$ is **conditionally convergent** if $\sum x_n$ is convergent but not absolutely convergent.

Thm $\sum x_n$ is absolutely convergent $\Rightarrow \sum x_n$ is convergent

Pf.: Let $\epsilon > 0$.

$\because \sum x_n$ is absolutely convergent,
i.e. $\sum |x_n|$ is convergent

$\therefore \exists M(\epsilon) \in \mathbb{N}$ s.t. if $m > n \geq M(\epsilon)$, then

$$\sum_{k=n+1}^m |x_k| < \epsilon$$

Letting $S_n = \sum_{k=1}^n x_k$, then $\forall m > n \geq M(\epsilon)$,

$$|S_m - S_n| = \left| \sum_{k=n+1}^m x_k \right| \leq \sum_{k=n+1}^m |x_k| < \epsilon$$

Cauchy's criterion implies that $\sum x_n$ is convergent. #

Grouping of series:

$$\sum x_n = x_1 + x_2 + \dots + x_n + \dots$$

$$= (x_1 + \dots + x_{k_1}) \xrightarrow{\text{def.}} y_1$$

$$+ (x_{k_1+1} + \dots + x_{k_2}) \xrightarrow{\text{def.}} y_2$$

+ ...

⋮

- (
- order of terms fixed
 - insert parentheses that group finite terms
-)

obtain: (y_k) .

Thm. Let $L = \sum x_n \in \mathbb{R}$ be convergent,
then the grouped series (y_k) is convergent
and it converges to L .

Pf. Set $S_n = \sum_{k=1}^n x_k$ to be the partial sum of $\sum x_n$

$\therefore \sum x_n = L$ is convergent

$\therefore \lim_{n \rightarrow \infty} S_n = L$

i.e. the sequence (S_n) converges to L .

Let $t_k = \sum_{i=1}^k y_i$ be the k^{th} partial sum of the grouped series $\sum y_k$.

then (t_k) is a subsequence of (S_n)

$\therefore \lim_{k \rightarrow \infty} t_k = L$

i.e. $\sum y_k$ converges to L . #

Def. (Rearrangement of Series)

$\sum y_k$ is a rearrangement of $\sum x_n$

if \exists a bijection $f: \mathbb{N} \rightarrow \mathbb{N}$ s.t. $y_k = x_{f(k)}, \forall k \in \mathbb{N}$.

Rk. Rearrangement may affect convergence.

e.g. (Riemann)

Let $\sum x_n$ be conditionally convergent and $c \in \mathbb{R}$, then \exists a rearrangement of $\sum x_n$ that converges to c .

Thm Let $\sum x_n$ be absolutely convergent in \mathbb{R} , then any rearrangement $\sum y_k$ of $\sum x_n$ converges to the same value.

Pf: Let $L = \sum x_n \in \mathbb{R}$, and let $\epsilon > 0$.

$\therefore \sum x_n$ is absolutely convergent

$\therefore \exists N \in \mathbb{N}$ s.t. if $n > N$ and $k > N$, then $|S_n - L| < \epsilon$ and $\sum_{k=N+1}^{\infty} |x_k| < \epsilon$.

Consider

$x_1 = y_{f^{-1}(1)}, x_2 = y_{f^{-1}(2)}, \dots, x_N = y_{f^{-1}(N)}$.

Define $M = \max(f^{-1}(1), \dots, f^{-1}(N)) \in \mathbb{N}$

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then all the terms x_1, \dots, x_N are contained in the partial sum $t_M \stackrel{\text{def.}}{=} y_1 + y_2 + \dots + y_M$

$$\therefore \forall n > N, \forall m \geq M, \underbrace{|t_m - s_n|}_{\text{contains finite terms excluding } x_1, \dots, x_N} \leq \sum_{k=N+1}^m |x_k| < \epsilon \quad (\text{for some } \delta > N)$$

$$\therefore \forall m \geq M,$$

$$|t_m - L| \leq |t_m - s_n| + |s_n - L| \quad (\text{let } n > N) \\ < \epsilon + \epsilon = 2\epsilon$$

$\therefore \epsilon > 0$ is arbitrary

$$\therefore \lim_{m \rightarrow \infty} t_m = L \text{ i.e. } \sum x_k \text{ converges to } L. \quad \#$$