Topic #12 Trigonometric Functions

We follow the same strategy as for introducing exponential / logarithm functions:

- Construct approximations sequences
- Uniform convergence on any bounded interval
- pointwise convergence
- Uniqueness (Taylor's Thm)
- More properties ...

Thm (Existence)
$\exists\ C:\mathbb{R}\to\mathbb{R}$ and $S:\mathbb{R}\to\mathbb{R}$ s.t.
(i) $C''(x) = -C(x)$, $S''(x) = -S(x)$, $\forall x\in\mathbb{R}$
(ii) $C(0) = 1$, $C'(0) = 0$; $S(0) = 0$, $S'(0) = 1$.

Pf: Step 1. Construct (C_n) and (S_n) such that
C_n, S_n are differentiable on $\mathbb{R}$ with
$$C_{n+1}' = -S_n,\quad S_n' = C_n,\quad n\in\mathbb{N}$$

In fact, define:
$$\begin{cases} C_1(x) = 1 \\ S_n(x) = \int_0^x C_n(t)\,dt \\ C_{n+1}(x) = 1 - \int_0^x S_n(t)\,dt \end{cases} \qquad x\in\mathbb{R},\ n\in\mathbb{N}.$$

$$\begin{array}{ccccccccccc} C_1 & & C_2 & & C_3 & \cdots & C_n & & C_{n+1} & \cdots \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & & \downarrow & \nearrow & \downarrow & \\ S_1 & & S_2 & & S_3 & \cdots & S_n & & S_{n+1} & \cdots \end{array}$$

the rest is left for your exercise. #

Step 2. Show (by induction):
$$C_{n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$$
$$S_{n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

(Exercise again). #

Step 3 Show: $\forall A>0$, (C_n), (S_n) are uniformly convergent on $[-A,A]$,
therefore, $\exists\, C:\mathbb{R}\to\mathbb{R}$ and $S:\mathbb{R}\to\mathbb{R}$ s.t.

$$C(x)=\lim_{n\to\infty}C_n(x),\quad \forall x\in\mathbb{R}$$

$$S(x)=\lim_{n\to\infty}S_n(x),\quad \forall x\in\mathbb{R}.$$

In fact, let $A>0$.

If $|x|\le A$ and $m>n>2A$

then
$$|C_m(x)-C_n(x)| = \left|\frac{x^{2n}}{(2n)!}-\frac{x^{2n+2}}{(2n+2)!}+\cdots\pm\frac{x^{2m-2}}{(2m-2)!}\right|$$

$$\le \frac{A^{2n}}{(2n)!}\left[1+\left(\frac{A}{2n}\right)^2+\cdots+\left(\frac{A}{2n}\right)^{2m-2n-2}\right]$$

$$\le \frac{A^{2n}}{(2n)!}\,\frac{1}{1-\frac{1}{16}} \qquad \left(\because \frac{A}{2n}<\frac{1}{4}\right)$$

$$1+\frac{1}{16}+\left(\frac{1}{16}\right)^2+\cdots+\left(\frac{1}{16}\right)^{m-n-1}$$

$$\to 0 \text{ as } n\to\infty$$

and

$$|S_m(x)-S_n(x)|$$

$$=\left|\int_0^x [C_m(t)-C_n(t)]\,dt\right|$$

$$\le \|C_m-C_n\|_{[-A,A]}\cdot A$$

$$\le \frac{A^{2n}}{(2n)!}\cdot\frac{16}{15}A \to 0 \text{ as } n\to\infty. \quad \#$$

step 4, show the rest:

- $C(0)=1$ and $S(0)=1$.

  In fact, $C_n(0)=1=S_n(0)$, $\forall n\in\mathbb{N}$

  $$\therefore\ C(0)=\lim_{n\to\infty}C_n(0)=1$$
  $$S(0)=\lim_{n\to\infty}S_n(0)=1$$

- $C'(x)=-S(x)$ and $S'(x)=C(x)$, $\forall x\in\mathbb{R}$

  In fact, $\because$ $C_n'=-S_{n-1}\rightrightarrows -S$ on $[-A,A]$

  $C_n\rightrightarrows C$ on $[-A,A]$ $\quad(A>0)$

  $\therefore$ C is differentiable on $[-A,A]$ with $C'=-S$ on $[-A,A]$

$\because A>0$ is arbitrary

$\therefore C$ is differentiable on $\mathbb{R}$ with $C' = -S$ on $\mathbb{R}$

Similarly

$\because S_n' = C_n \rightrightarrows C$ on $[-A, A]$

$S_n \rightrightarrows S$ on $[-A, A]$ $(\forall A>0)$

$\therefore S$ is differentiable on $\mathbb{R}$ with $S' = C$ on $\mathbb{R}$

- Direct to see:

$$C'(0) = -S(0) = -1; \quad S'(0) = C(0) = 1$$

$$C''(x) = (-S(x))' = -S'(x) = -C(x), \quad \forall x \in \mathbb{R}$$

$$S''(x) = (C(x))' = -C'(x) = S(x), \quad \forall x \in \mathbb{R}. \quad \#\#$$

Corollary It further holds:

(iii) $C'(x) = -S(x)$, $S'(x) = C(x)$, $\forall x \in \mathbb{R}$;

and C and S have derivatives of all orders.

(iv) (Pythagorean Identity):

$$(C(x))^2 + (S(x))^2 = 1, \quad \forall x \in \mathbb{R}.$$

pf: (iii) direct.

(iv) Let $f(x) \overset{\text{def.}}{=} (C(x))^2 + (S(x))^2$, $x \in \mathbb{R}$, then

$$\begin{aligned} f'(x) &= 2C(x)\,C'(x) + 2S(x)\,S'(x) \\ &= 2C(x)[-S(x)] + 2S(x)[C(x)] \\ &= 0, \quad \forall x \in \mathbb{R} \end{aligned}$$

$\therefore f \equiv$ const on $\mathbb{R}$

$\because f(0) = (C(0))^2 + (S(0))^2 = 1^2 + 0^2 = 1$

$\therefore f(x) \equiv 1$ on $\mathbb{R}$. #

Thm (Uniqueness)

C and S satisfying (i) & (ii) are unique.

pf: Let $C_1: \mathbb{R} \to \mathbb{R}$, $C_2: \mathbb{R} \to \mathbb{R}$ be functions such that $\begin{cases} C_i''(x) = -C_i(x), & \forall x \in \mathbb{R}, i=1,2, \\ C_i(0) = 1, \; C_i'(0) = 0, & i=1,2. \end{cases}$

Define $D(x) = C_1(x) - C_2(x)$, $x \in \mathbb{R}$,

then $\begin{cases} D''(x) = -D(x), \ \forall x \in \mathbb{R} \\ D(0) = C_1(0) - C_2(0) = 1 - 1 = 0 \\ D'(0) = C_1'(0) - C_2'(0) = 0 - 0 = 0 \end{cases}$

By Induction, $D^{(k)}(0) = 0$, $\forall k \in \mathbb{N}$.

Let $0 \neq x \in \mathbb{R}$ and $I_x = [0, x]$ if $x > 0$ and $[x, 0]$ if $x < 0$.

Apply Taylor's Thm to D on I_x; $\forall n \in \mathbb{N}$,

$$D(x) = D(0) + \frac{D'(0)}{1!}x + \cdots + \frac{D^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{D^{(n)}(c_n)}{n!}x^n$$

for some $c_n \in I_x$

$$= \frac{D^{(n)}(c_n)}{n!}x^n$$

indep't of k

Observe: $D^{(k)}(y) = \pm D(y)$ or $\pm(S_1 - S_2)(y)$, $\forall y \in \mathbb{R}$, $\forall k \in \mathbb{N}$ (Exercise)

that are continuous on $\mathbb{R}$.

then $\exists K_x > 0$, independent of n s.t.

$$\|D^{(n)}\|_{I_x} \leq \max\left(\|D\|_{I_x}, \|S_1 - S_2\|_{I_x}\right) \leq K_x, \quad \forall n \in \mathbb{N}.$$

therefore

$$|D(x)| \leq \frac{|D^{(n)}(c_n)|}{n!}|x|^n \leq \frac{K_x |x|^n}{n!} \longrightarrow 0 \text{ as } n \to \infty$$

then $D(x) = 0$, Thus $D(x) = 0$, $\forall x \in \mathbb{R}$,

i.e. $C_1(x) - C_2(x) = 0$, $\forall x \in \mathbb{R}$.

Applying the same argument,

if $S_1 : \mathbb{R} \to \mathbb{R}$, $S_2 : \mathbb{R} \to \mathbb{R}$ are functions s.t. $\begin{cases} S_i''(x) = -S_i(x), \ \forall x \in \mathbb{R} \\ S_i(0) = 0, \ S_i'(0) = 0, \end{cases}$ for $i = 1, 2$,

then $S_1(x) = S_2(x)$, $\forall x \in \mathbb{R}$. #

Def. Such unique functions $C : \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{cases} \text{(i) } C''(x) = -C(x), \quad S''(x) = -S(x), \ \forall x \in \mathbb{R} \\ \text{(ii) } C(0) = 1, \ C'(0) = 0; \quad S(0) = 0, \ S'(0) = 1, \end{cases}$$

are called the cosine function and the sine function, resp. #

We write: $\cos x = C(x)$, $\sin x = S(x)$, $x \in \mathbb{R}$.

Thm If $f: \mathbb{R} \to \mathbb{R}$ satisfies
$$f''(x) = -f(x), \quad \forall x \in \mathbb{R}$$
then $\exists \alpha, \beta \in \mathbb{R}$ s.t.
$$f(x) = \alpha C(x) + \beta S(x), \quad \forall x \in \mathbb{R}$$

Pf: Define
$$g(x) = f(0) C(x) + f'(0) S(x), \quad x \in \mathbb{R}$$
Check that

① $g''(x) = -g(x)$

② $g(0) = f(0)$.

Furthermore

$\because g'(x) = -f(0) S(x) + f'(0) C(x)$

$\therefore g'(0) = f'(0)$

Now, define
$$h(x) = f(x) - g(x), \quad x \in \mathbb{R}$$
then
$$h''(x) = -h(x), \; x \in \mathbb{R} \quad \text{with} \quad \begin{cases} h(0) = f(0) - g(0) = 0 \\ h'(0) = f'(0) - g'(0) = 0 \end{cases}$$

By Uniquess of solutions,
$$h(x) = 0, \quad \forall x \in \mathbb{R}$$
i.e. $f(x) = g(x), \; \forall x \in \mathbb{R}$. #

More basic properties of cosine and sine functions

Thm it holds:

(V) C is even and S is odd: $C(-x) = C(x)$, $S(-x) = S(x)$, $\forall x \in \mathbb{R}$

(Vi) (Addition formulas): $\forall x, y \in \mathbb{R}$
$$C(x+y) = C(x)C(y) - S(x)S(y)$$
$$S(x+y) = S(x)C(y) + C(x)S(y).$$

Pf (V): Define $\varphi(x) = C(-x) - C(x)$.

Verify $\varphi''(x) = -\varphi(x)$ with $\begin{cases} \varphi(0) = 0 \\ \varphi'(0) = 0 \end{cases}$

By Uniquess, $\varphi(x) \equiv 0$, i.e. $C(-x) = C(x)$, $\forall x \in \mathbb{R}$.
Similarly, we have: $S(-x) = -S(x)$, $\forall x \in \mathbb{R}$. #

(vi) Fix $y \in \mathbb{R}$.

Define $f(x) = C(x+y)$, $\forall x \in \mathbb{R}$

Verify $f''(x) = -f(x)$, $\forall x \in \mathbb{R}$

then $\exists \alpha, \beta$ (may depend on y) $\in \mathbb{R}$ s.t.

$$f(x) = C(x+y) = \alpha C(x) + \beta S(x), \quad \forall x \in \mathbb{R}$$

Take derivative in x:

$$f'(x) = -S(x+y) = -\alpha S(x) + \beta C(x), \quad \forall x \in \mathbb{R}$$

Setting $x=0$,

$$C(y) = \alpha \cdot 1 + \beta \cdot 0 = \alpha$$
$$-S(y) = -\alpha \cdot 0 + \beta \cdot 1 = \beta$$

this proves:

$$f(x) = C(x+y) = C(x)C(y) - S(x)S(y).$$

For the 2nd formula, the proof is similar. ##

Thm For $x \geq 0$, it holds:

(vii) $-x \leq S(x) \leq x$;

(viii) $1 - \frac{1}{2}x^2 \leq C(x) \leq 1$;

(ix) $x - \frac{1}{6}x^3 \leq S(x) \leq x$;

(x) $1 - \frac{1}{2}x^2 \leq C(x) \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$.

Pf: Omitted.

Now, we explain how to construct π.

Lemma
- $\exists$ a root γ of $C(x)$ in $[\sqrt{2}, \sqrt{3})$
- Moreover, $C(x) > 0$, $\forall x \in [0, \gamma)$
- 2γ is the smallest positive root of $S(x)$.

Pf: Recall

$$1-\tfrac{1}{2}x^2 \le C(x) \le 1-\tfrac{1}{2}x^2+\tfrac{1}{24}x^4, \quad x \ge 0.$$

Use it to check:

$$C(\sqrt{2}) \ge 1-\tfrac{1}{2}(\sqrt{2})^2 = 0$$

$$C(\sqrt{3}) \le 1-\tfrac{1}{2}(\sqrt{3})^2+\tfrac{1}{24}(\sqrt{3})^4$$

$$= 1-\tfrac{3}{2}+\tfrac{9}{24}$$

$$= \frac{24-36+9}{24} = -\frac{3}{24} = -\frac{1}{8} < 0$$

Intermediate Value Thm gives:

$$\exists\, x \in [\sqrt{2}, \sqrt{3}) \text{ s.t. } C(x)=0$$

Let γ be the smallest such root of $C(x)$ in $[\sqrt{2}, \sqrt{3})$

i.e. $\gamma = \min\{x \in [\sqrt{2}, \sqrt{3}) : C(x)=0\}$

Observe:

if $x \in [\sqrt{2}, \gamma)$ then $C(x) \ne 0$ by def of γ

if $x \in [0, \sqrt{2})$, then $C(x) \ge 1-\tfrac{1}{2}x^2 > 0$

Therefore continuity of $C(x)$ gives:

$$C(x) > 0, \quad \forall x \in [0, \gamma)$$

(Exercise!)

It remains to show: 2γ is smallest positive root of S.

In fact:

note $S(2x) = 2S(x)C(x)$

then $S(2\gamma) = 2S(\gamma)C(\gamma) = 0$

i.e. 2γ is a positive root of S.

Observe: If $2\delta > 0$ is the smallest positive root of S,

then $S(2\delta) = 2S(\delta)C(\delta) = 0$ gives: $C(\delta) = 0$

$\because \gamma$ is the smallest positive root of C $\quad$ ($\therefore \gamma \le \delta$ i.e. $2\gamma \le 2\delta$)

$\therefore \delta = \gamma$. #

Def. $\pi \overset{\text{def.}}{=} 2\gamma$ is the smallest positive root of S

$\boxed{\begin{array}{l} \textbf{Thm}\text{ It holds:} \\ \text{(xi) } C, S \text{ are } 2\pi\text{-periodic:} \\ \qquad C(x+2\pi) = C(x), \quad S(x+2\pi) = S(x), \quad \forall x \in \mathbb{R} \\ \text{(xii) } \begin{cases} C(\frac{\pi}{2}-x) = S(x), & C(\frac{\pi}{2}+x) = -S(x), \\ S(\frac{\pi}{2}-x) = C(x), & S(\frac{\pi}{2}+x) = C(x), \end{cases} \quad \forall x \in \mathbb{R}. \end{array}}$

Pf: Omitted.