Topic#11 Exponential and Logarithmic Functions

Keywords:

* Constructive proof for existence
* Uniqueness
* Basic properties
* Logarithm function as inverse of exponential function

Exponential Function

Thm (Existence and uniqueness)
$\exists!\, E:\mathbb{R}\to\mathbb{R}$ s.t.
(i) $E'(x) = E(x), \ \forall x\in\mathbb{R}$
(ii) $E(0) = 1$.

Proof (Lengthy)

part I Existence

step 1. Construct an approximation sequence (E_n):

Define:

$$E_1(x) = 1 + x$$

$$(*) \qquad E_{n+1}(x) = 1 + \int_0^x E_n(t)\,dt, \quad n = 1, 2, \dots$$

Claim: (E_n) is well-defined and for each $n\in\mathbb{N}$, E_n is differentiable on $\mathbb{R}$ with

$$E_{n+1}'(x) = E_n(x), \quad \forall x\in\mathbb{R}.$$

Pf: Induction on $n\geq 1$:

$n=1$: $E_1(x) = 1+x$ is differentiable on $\mathbb{R}$

Assume: E_n is differentiable on $\mathbb{R}$ for $n\geq 1$

then E_n is continuous on $\mathbb{R}$,

and hence intergrable on any bounded interval.

$\therefore$ By $(*)$, E_{n+1} is well-defined

and Fundamental Thm (2^{nd} Form) tells:

E_{n+1} is differentiable at each $x\in\mathbb{R}$

with $E_{n+1}'(x) = E_n(x), \ \forall x\in\mathbb{R}$. #

Step 2. Compute the explicit form of (E_n):

$\boxed{\text{Claim: } E_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}, \quad x \in \mathbb{R} \text{ for } n \in \mathbb{N}.}$

Pf: Induction on $n \geq 1$ again. Exercise. #

Step 3. Determine the limit of (E_n):

$\boxed{\text{Claim: } \exists E : \mathbb{R} \to \mathbb{R} \text{ with } E(0) = 1 \text{ s.t. } E_n \to E \text{ on } \mathbb{R}.}$

Pf: Let $A > 0$,

to show: (E_n) is uniformly convergent on $[-A, A]$.

In fact, let $|x| \leq A$ and $m > n > 2A$, then

$$
\begin{aligned}
&|E_m(x) - E_n(x)| \\
&= \left| \frac{x^{n+1}}{(n+1)!} + \cdots + \frac{x^m}{m!} \right| \\
&\leq \frac{|x|^{n+1}}{(n+1)!} + \cdots + \frac{|x|^m}{m!} \\
&\leq \frac{A^{n+1}}{(n+1)!} + \cdots + \frac{A^m}{m!} \\
&= \frac{A^{n+1}}{(n+1)!} \left[1 + \frac{A}{n+2} + \cdots + \frac{A^{m-n-1}}{(n+2)(n+3)\cdots m} \right] \\
&< \frac{A^{n+1}}{(n+1)!} \left[1 + \frac{A}{n} + \cdots + \left(\frac{A}{n}\right)^{m-n-1} \right] \\
&< \frac{A^{n+1}}{(n+1)!} \left[1 + \left(\tfrac{1}{2}\right) + \cdots + \left(\tfrac{1}{2}\right)^{m-n-1} \right] \quad \left(\because \frac{A}{n} < \frac{1}{2}\right) \\
&< \frac{A^{n+1}}{(n+1)!} \, 2 \longrightarrow 0 \quad \left(\because \lim_{n\to\infty} \frac{A^n}{n!} = 0\right)
\end{aligned}
$$

$\therefore$ (E_n) is uniformly convergent on $[-A, A]$.

$\because$ $A > 0$ is arbitrary

$\therefore$ $(E_n(x))$ converges for each $x \in \mathbb{R}$

We then define $E(x) = \lim_{n\to\infty} E_n(x), \ \forall x \in \mathbb{R}$.

$\because$ $E_n(0) = 1, \ \forall n \in \mathbb{N}$

$\therefore$ $E(0) = \lim_{n\to\infty} E_n(0) = 1$. #

Step 4 $\boxed{\text{Show: } E \text{ is differentiable on } \mathbb{R} \text{ with } E'(x)=E(x),\ \forall x\in\mathbb{R}.}$

In fact, recall that:

$$\forall A>0,\quad E_n \rightrightarrows E \text{ on } [-A,A]$$
$$E_n' = E_{n-1} \rightrightarrows E \text{ on } [-A,A]$$

Then, by Thm (Interchange of Limit and Derivative),

E is differentiable on $[-A,A]$ with $E'=E$ on $[-A,A]$.

$\because A>0$ is arbitrary

$\therefore E$ is differentiable on $\mathbb{R}$ with $E'=E$ on $\mathbb{R}$.

Part II: Uniqueness

Let $E_*:\mathbb{R}\to\mathbb{R}$ be another function s.t. $\begin{cases} E_*'(x)=E_*(x), \ \forall x\in\mathbb{R} \\ E_*(0)=1. \end{cases}$

Define $F=E-E_*$, to show: $F\equiv 0$ on $\mathbb{R}$.

Step 1 Observe: $F^{(n)}=F$ on $\mathbb{R}$ and $F(0)=0$.

In fact, obvious to see: $F(0)=E(0)-E_*(0)=1-1=0$.

To show $F^{(n)}=F$ on $\mathbb{R}$ for any $n\in\mathbb{N}$, use induction. (Ex.)

Step 2 Let $0\neq x\in\mathbb{R}$, to show: $F(x)=0$.

In fact, let $I=[0,x]$ if $x>0$ and $[x,0]$ if $x<0$.

Applying Taylor's Thm, $\forall n\in\mathbb{N}$, $\exists\, c_n\in I$ s.t.

$$F(x) = F(0) + \frac{F'(0)}{1!}x + \cdots + \frac{F^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{F^{(n)}(c_n)}{n!}x^n$$

Observe: ① $\forall k\in\mathbb{N}$, $F^{(k)}(0)=F(0)=0$

② $\because F$ continuous on I

$\therefore F$ bounded on I,

i.e. $\exists K=K(x)>0$ s.t. $\|F\|_I\le K$

then $|F^{(n)}(c_n)| = |F(c_n)| \le \|F\|_I \le \underbrace{K}_{\text{indep't of } n}$

Plug back,

$$|F(x)| = \left|\frac{F(c_n)}{n!}x^n\right| \le K\frac{|x|^n}{n!} \longrightarrow 0 \text{ as } n\to\infty.$$

##

Basic property: $E(x) > 1+x, \ \forall x>0.$

Pf: From the explicit formula of E_n, we see:

$\forall x>0$, $(E_n(x))$ is strictly increasing

$\therefore \ E(x) > E_1(x), \ \forall x>0$

i.e. $E(x) > 1+x, \ \forall x>0.$ #

Def. (Exponential Function)

Such unique function $E: \mathbb{R}\to\mathbb{R}$ satisfying

$$\begin{cases} \text{(i)} \ E'(x) = E(x), \ \forall x\in\mathbb{R} \\ \text{(ii)} \ E(0) = 1 \end{cases}$$

is called the exponential function and

$$e := E(1) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k!} \quad \left(= \sum_{k=1}^{\infty} \frac{1}{k!} \right)$$

is called the Euler's number.

We write:

$$\exp(x) := E(x) \quad \text{or} \quad e^x = E(x).$$

Thm It further holds:

(iii) $E(x) \neq 0, \ \forall x\in\mathbb{R}$

(iv) $E(x+y) = E(x)E(y), \ \forall x,y\in\mathbb{R}$

(v) $E(r) = e^r, \ \forall r\in\mathbb{Q}$.

(vi) E is strictly increasing on $\mathbb{R}$ with

$$\begin{cases} \bullet \ \operatorname{Rang} E = \{y\in\mathbb{R} : y>0\} \\ \bullet \ \lim_{x\to-\infty} E(x) = 0 \\ \bullet \ \lim_{x\to\infty} E(x) = \infty. \end{cases}$$

Pf: (iii): Otherwise $\exists \alpha\in\mathbb{R}$ s.t. $E(\alpha)=0$.

Note: $E(0)=1$, $\therefore \alpha\neq 0$.

Let $I = [0,\alpha]$ if $\alpha>0$ or $[\alpha,0]$ if $\alpha<0$.

Let $K>0$ be such that $\|E\|_I \leq K$,

Similarly as before, apply Taylor's Thm:

$$1 = E(0) = E(\alpha) + \frac{E'(\alpha)}{1!}(-\alpha) + \dots + \frac{E^{(n-1)}(\alpha)}{(n-1)!}(-\alpha)^{n-1} + \frac{E^{(n)}(c_n)}{n!}(-\alpha)^n$$

$$E^{(n)}(x) = E(x)$$

then $$1 = \frac{E(c_n)}{n!}(-x)^n \le \frac{|E(c_n)|}{n!}|x|^n \le K\frac{|x|^n}{n!} \to 0$$

which is a contradiction. #

(iv): Fix $y \in \mathbb{R}$ and define
$$G(x) = \frac{E(x+y)}{E(y)}, \quad x \in \mathbb{R} \qquad (\text{note: } E(y) \neq 0 \text{ by (iii)})$$

Compute:
$$G(0) = \frac{E(0+y)}{E(y)} = \frac{E(y)}{E(y)} = 1$$

$$G'(x) = \frac{E'(x+y)}{E(y)} = \frac{E(x+y)}{E(y)} = G(x), \quad \forall x \in \mathbb{R}$$

By Uniqueness,
$$G(x) = E(x), \quad \forall x \in \mathbb{R},$$
i.e. $$\frac{E(x+y)}{E(y)} = E(x).$$
i.e. $E(x+y) = E(x)E(y)$. #

(v): By (iv) and induction,
$$E(nx) = (E(x))^n, \quad \forall x \in \mathbb{R}, \quad n = 1, 2, \cdots$$
then
$$e = E(1) = E\left(n \cdot \tfrac{1}{n}\right) = \left(E\left(\tfrac{1}{n}\right)\right)^n$$
$$\therefore E\left(\tfrac{1}{n}\right) = e^{1/n}.$$
Note: $\forall m \in \mathbb{N}, \forall x \in \mathbb{R}$
$$E(-mx) = \frac{1}{E(mx)} = \frac{1}{(E(x))^m} = (E(x))^{-m}$$

then $\forall m \in \mathbb{Z}, \forall n \in \mathbb{N}$,
$$E\left(\frac{m}{n}\right) = \left(E\left(\tfrac{1}{n}\right)\right)^m = \left(e^{\frac{1}{n}}\right)^m = e^{\frac{m}{n}}. \quad \#$$

(vi): $\because$ $E(0) = 1 > 0$; $E(x) \neq 0, \forall x \in \mathbb{R}$; E continuous on $\mathbb{R}$

$\therefore$ Intermediate Value Thm gives: $E(x) > 0, \forall x \in \mathbb{R}$. (Exercise)

$\therefore E'(x) = E(x) > 0,\ \forall x \in \mathbb{R}$

i.e. E is strictly increasing on $\mathbb{R}$.

Moreover

- $\because E(x) > 1 + x,\ \forall x > 0$

  $\therefore \lim_{x \to \infty} E(x) = \infty$

- $\because \forall x > 0,\ 0 < E(-x) = \frac{1}{E(x)}$ $\left\{ \text{then } \lim_{x\to\infty} E(-x) = 0 \right\}$

  $\therefore \lim_{x \to -\infty} E(x) = 0$

- $\forall y > 0$, Intermediate Value Thm gives that:

  $\exists x \in \mathbb{R}$ s.t. $y = E(x)$.

  (Why? Your exercise!).

  $\left[\begin{array}{l} \exists -\infty < \epsilon < M < \infty \\ \text{s.t. } E(\epsilon) < y < E(M) \end{array} \right]$ ##

Logarithm Function

Def $L(x) \overset{\text{def.}}{=} E^{-1}(x)$: inverse to $E : \mathbb{R} \to \mathbb{R}$

called the (natural) logarithm.

Often write: $L(x) = \ln x$.

RK: By def.,

$L \circ E(x) = x,\ \forall x \in \mathbb{R}$; i.e., $\ln e^x = x,\ \forall x \in \mathbb{R}$

$E \circ L(y) = y,\ \forall y > 0$; i.e. $e^{\ln y} = y,\ \forall y > 0$.

Thm: L is strictly increasing with Domain$(L) = \{x \in \mathbb{R} : x > 0\}$

Range$(L) = \mathbb{R}$,

and

(vii) $L'(x) = \frac{1}{x},\ \forall x > 0$

(viii) $L(xy) = L(x) + L(y),\ \forall x > 0, \forall y > 0$.

(ix) $L(1) = 0,\ L(e) = 1$

(x) $L(x^r) = r L(x),\ \forall x > 0,\ \forall r \in \mathbb{Q}$

(xi) $\lim_{x \to 0+} L(x) = -\infty,\ \lim_{x \to \infty} L(x) = \infty$.

Pf: $\because$ E is strictly increasing with Dom $= \mathbb{R}$ and Range $= \{y \in \mathbb{R};\ y > 0\}$

$\therefore$ The inverse L to E is strictly increasing with Domain $= \{x \in \mathbb{R} : x > 0\}$ and Range $= \mathbb{R}$.

(vii) $\because$ $E'(x) = E(x) > 0$

$\therefore$ By Differentiability Thm for Inverse Function

L is differentiable on $(0, \infty)$ with

$$L'(x) = \frac{1}{(E' \circ L)(x)} = \frac{1}{E \circ L(x)} = \frac{1}{x}, \quad \forall x > 0.$$

(viii) Let $x > 0$, $y > 0$.

Denote $u = L(x)$, $v = L(y)$

i.e. $x = E(u)$, $y = E(v)$

then $xy = E(u)E(v) = E(u+v)$

$\therefore$ $L(xy) = (L \circ E)(u+v) = u + v = L(x) + L(y)$.

(ix) Consequence of $E(0) = 1$ and $E(1) = e$.

(x) Exercise: Follow similar argument as in (v).

(xi) $\because$ $e = E(1) > 1 + 1 = 2$

$\therefore$ $\lim\limits_{n \to \infty} e^n = \infty$, $\lim\limits_{n \to \infty} e^{-n} = 0$

then $\lim\limits_{x \to \infty} L(x) = \lim\limits_{n \to \infty} L(e^n) = \lim\limits_{n \to \infty} n = \infty$

$\lim\limits_{x \to -\infty} L(x) = \lim\limits_{n \to \infty} L(e^{-n}) = \lim\limits_{n \to \infty} (-n) = -\infty$. ##

Power Functions

Recall: Function "$x \mapsto x^r$, $x>0$" was defined before for $r \in \mathbb{Q}$.

Def. Let $\alpha \in \mathbb{R}$ and $x>0$, then

$$x^\alpha \overset{\text{def.}}{=} e^{\alpha \ln x} = E(\alpha L(x))$$

and $x \mapsto x^\alpha$ $(x>0)$ is called the power function with exponent α.

Thm If $\alpha \in \mathbb{R}$ and $x, y \in (0,\infty)$ then

(a) $1^\alpha = 1$

(b) $x^\alpha > 0$

(c) $(xy)^\alpha = x^\alpha y^\alpha$

(d) $\left(\frac{x}{y}\right)^\alpha = \frac{x^\alpha}{y^\alpha}$

Moreover, if $\alpha, \beta \in \mathbb{R}$ and $x \in (0,\infty)$, then

(e) $x^{\alpha+\beta} = x^\alpha y^\beta$

(f) $(x^\alpha)^\beta = x^{\alpha\beta}$

(h) $x^{-\alpha} = \frac{1}{x^\alpha}$

(i) if $\alpha<\beta$ then $x^\alpha < x^\beta$, $\forall x>1$.

Thm Let $\alpha \in \mathbb{R}$, then $x \mapsto x^\alpha$ $(x>0)$ is continuous and differentiable with $Dx^\alpha = \alpha x^{\alpha-1}$, $\forall x>0$.

Pf. Chain Rule gives

$$\begin{aligned} Dx^\alpha &= De^{\alpha \ln x} \\ &= e^{\alpha \ln x} D(\alpha \ln x) \\ &= e^{\alpha \ln x} \cdot \alpha \frac{1}{x} \\ &= x^\alpha \cdot \frac{\alpha}{x} \\ &= \alpha x^{\alpha-1}, \quad \forall x>0. \quad \# \end{aligned}$$

Def: Let $a > 0$ with $a \neq 1$, then

$$\log_a(x) \overset{\text{def.}}{=} \frac{\ln x}{\ln a}, \quad \forall x > 0$$

called the logarithm of x to the base a. #

Rks: ① if $a = e$, then $\log_a(x) = \frac{\ln x}{\ln e} = \ln x$ is the (natural) logarithm.

② properties of $\log_a(x)$ $(x > 0)$: refer to exercises. ##