Topic #10: Interchange of Limits

Examples:

(a) Recall: $g_n(x) = x^n \longrightarrow g(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$ on $[0,1]$ (pointwise)

(continuous on $[0,1]$) (discontinuous at $x=1$)

$$\lim_{n\to\infty} \lim_{x\to 1} g_n(x) = \lim_{n\to\infty} g_n(1) = 1$$

$$\lim_{x\to 1} \lim_{n\to\infty} g_n(x) = \lim_{\substack{x\to 1 \\ (x\neq 1)}} g(x) = 0$$

$$\therefore \lim_{n\to\infty} \lim_{x\to 1} g_n(x) \neq \lim_{x\to 1} \lim_{n\to\infty} g_n(x)$$

Can't change limits for "pointwise convergence"

("Pointwise limit" of sequence of continuous functions may NOT be continuous!)

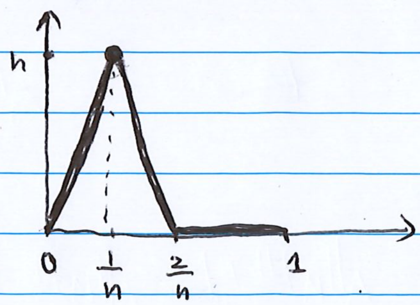
(b) The same example:

$$g_n'(x) = n x^{n-1}, \quad 0 \le x \le 1$$

$$g'(x) = \begin{cases} 0, & 0 \le x < 1 \\ \text{does NOT exist} & x = 1 \end{cases}$$

"pointwise limit" of sequence of differentiable functions may NOT be continuous!

(c) $f_n(x) = \begin{cases} n^2 x, & 0 \le x < \frac{1}{n} \\ -n^2(x - \frac{2}{n}), & \frac{1}{n} \le x < \frac{2}{n} \\ 0, & \frac{2}{n} \le x \le 1 \end{cases}$, $0 \le x \le 1$; $n \ge 2$.



Note:
- f_n continuous on $[0,1]$
- $f_n(x) \to f(x) \equiv 0, \ \forall x \in [0,1]$

(if $x=0$, then $f_n(0)=0$.
if $0 < x \le 1$, then for n large enough, s.t. $\frac{2}{n} \le x$, then $f_n(x) = 0$.

$\because f_n$ continuous on $[0,1]$

$\therefore f_n$ Riemann integrable with $\int_0^1 f_n = \frac{1}{2}\cdot\frac{2}{n}\cdot n = 1$

$\therefore \lim_{n\to\infty}\int_0^1 f_n = 1 \neq 0 = \int_0^1 f$

namely, $\lim_{n\to\infty}\int_0^1 f_n \neq \int_0^1 \lim_{n\to\infty} f_n$

limit of integrals $\neq$ integral of "pointwise limit"

(d) $h_n(x) \overset{\text{def.}}{=} 2nx\,e^{-nx^2}$, $0 \le x \le 1$.

then $\int_0^1 h_n = \int_0^1 2nx\,e^{-nx^2}\,dx = \int_0^1 (-e^{-nx^2})'\,dx$

$= -e^{-nx^2}\Big|_0^1 = 1 - e^{-n}$

$\therefore \lim_{n\to\infty}\int_0^1 h_n = 1$

And, $\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} 2nx\,e^{-nx^2} = 0$, $\forall x \in [0,1]$. (Exercise!)

Thus,

$$\lim_{n\to\infty}\int_0^1 h_n \neq \int_0^1 \lim_{n\to\infty} h_n$$

Question: Under what conditions, the limit of a convergent sequence of continuous [resp., differentiable, integrable] functions can be still continuous [resp. differentiable, integrable]?

1st issue: Interchange of Limit and Continuity

Thm If $f_n \rightrightarrows f$ on A where (f_n) is a sequence of continuous fns on A, then the limit function f is continuous on A.

$(\frac{\epsilon}{3})$-strategy

Pf: Let $c \in A$, then $\forall x \in A$, $\forall n \in \mathbb{N}$,

$$|f(x) - f(c)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)|$$

$$\le 2\sup_{x\in A}|f(x) - f_n(x)| + |f_n(x) - f_n(c)|$$

$$= 2\|f_n - f\|_A + |f_n(x) - f_n(c)| \quad \text{———} \quad (*)$$

Let $\epsilon > 0$.

$\because f_n \rightrightarrows f$ on A

$\therefore \exists H = H(\frac{\epsilon}{3}) \in \mathbb{N}$ s.t. $\forall n \geq H$, $\|f_n - f\|_A < \frac{\epsilon}{3}$.

in particular, $\|f_H - f\|_A < \frac{\epsilon}{3}$.

$\because f_H$ is continuous on A

$\therefore \lim_{x \to c} f_H(x) = f_H(c)$,

i.e. $\exists \delta = \delta(\frac{\epsilon}{3}, c, f_H)$ s.t. if $|x - c| < \delta$,

then $|f_H(x) - f_H(c)| < \frac{\epsilon}{3}$.

Therefore, by $(*)$, if $|x-c| < \delta$, then

$$|f(x) - f(c)| \leq 2\|f_H - f\|_A + |f_H(x) - f_H(c)|$$

$$< \frac{2}{3}\epsilon + \frac{1}{3}\epsilon = \epsilon.$$

Thus, f is continuous at c.

$\because c \in A$ is arbitrary

$\therefore f$ is continuous on A. #

RK: "Unifor Convergence" is a sufficient condition, but may not be necessary. #

(Exercise 2)

$\boxed{2^{nd} \text{ issue}}$: Interchange of Limit and Derivative

Thm. Let
- $I \subseteq \mathbb{R}$ be a bounded interval;
- $f_n : I \to \mathbb{R}$ sequence of functions;
- $\exists x_0 \in I$ s.t. $f_n(x_0)$ converges as $n \to \infty$;
- f_n' exist on I, $\forall n \in \mathbb{N}$;
- $f_n' \rightrightarrows g$ on I for a function $g : I \to \mathbb{R}$;

Four cases: $[a,b]$, $(a,b]$, $[a,b)$, (a,b), $-\infty < a < b < \infty$

then

$\exists$ differentiable $f : I \to \mathbb{R}$ with $f' = g$ on I s.t.

$f_n \rightrightarrows f$ on I

(long proof; by two steps)

Pf: step 1. Find f via Cauchy Criterion for uniform convergence.

Claim: $\exists f$ continuous on I s.t. $f_n \rightrightarrows f$ on I.

In fact, let $m, n \in \mathbb{N}$ and $x \in I$.

Mean Value Thm gives:

$$(f_m - f_n)(x) - (f_m - f_n)(x_0) = (f_m - f_n)'(y)\,(x - x_0)$$

for some y between x and x_0.

then $|f_m(x) - f_n(x)| \leq |f_m(x_0) - f_n(x_0)| + |f_m'(y) - f_n'(y)| \cdot |x - x_0|$

$\leq |f_m(x_0) - f_n(x_0)| + \|f_m' - f_n'\|_I \, (b-a)$

($\because$ $a<b$ are the endpts of interval I)

Since $x \in I$ is arbitrary,

Key (*)

$$\|f_m - f_n\|_I = \sup_{x\in I} |f_m(x) - f_n(x)| \leq |f_m(x_0) - f_n(x_0)| + \|f_m' - f_n'\|_I (b-a)$$

Let $\epsilon > 0$.

① $\because$ $(f_n(x_0))$ is convergent in $\mathbb{R}$

$\therefore$ Cauchy Criterion for convergence of sequence of numbers tells: $\exists H_1 = H_1(\frac{\epsilon}{2}) \in \mathbb{N}$ s.t. $\forall m, n \geq H_1$,

$$|f_m(x_0) - f_n(x_0)| < \frac{\epsilon}{2}$$

② $\because$ (f_n') is uniformly convergent on I

$\therefore$ Cauchy Criterion for uniform convergence of sequence of functions tells:

$\exists H_2 = H_2\left(\frac{\epsilon}{2(b-a)}\right) \in \mathbb{N}$ s.t. $\forall m, n \geq H_2$,

$$\|f_m' - f_n'\|_I < \frac{\epsilon}{2(b-a)}$$

Define $H = \max\{H_1, H_2\} \in \mathbb{N}$,

then $\forall m, n \geq H$, applying ① & ② to (*),

$$\|f_m - f_n\|_I < \frac{\epsilon}{2} + \frac{\epsilon}{2(b-a)} \cdot (b-a) = \epsilon.$$

By Cauchy Criterion again, (f_n) is uniformly convergent on I.

$\therefore$ $\exists f: I \to \mathbb{R}$ s.t. $f_n \rightrightarrows f$ on I.

By Thm for Interchange of limit and continuity, since f_n's are continuous on I, the limit fn. f is continuous on I. #

Step 2. Show f is differentiable with $f' = g$.

claim: f is differentiable on I with $f' = g$ on I.

Let $c \in I$, then $\forall x \in I$ with $x \neq c$, $\forall m, n \in \mathbb{N}$,

$$(f_m - f_n)(x) - (f_m - f_n)(c) \xlongequal{\text{Mean-Value Thm}} (f_m - f_n)'(z)(x-c)$$

for some z between x and c.

$$\therefore \left| \frac{f_m(x) - f_m(c)}{x-c} - \frac{f_n(x) - f_n(c)}{x-c} \right| = |f_m'(z) - f_n'(z)|$$

$$\leq \|f_m' - f_n'\|_I$$

then $\forall \epsilon > 0$, $\forall m, n \geq H_1$,

$$\left| \frac{f_m(x) - f_m(c)}{x-c} - \frac{f_n(x) - f_n(c)}{x-c} \right| < \frac{\epsilon}{2(b-a)}.$$

Letting $m \to \infty$ and $f_m \rightrightarrows f$ on I, we have: for $x \neq c$,

$$\left| \frac{f(x) - f(c)}{x-c} - \frac{f_n(x) - f_n(c)}{x-c} \right| < \frac{\epsilon}{2(b-a)}, \quad \forall n \geq H_1.$$

$\because \lim_{n\to\infty} f_n'(c) = g(c)$

$\therefore$ for the same $\epsilon > 0$, $\exists N = N(\epsilon) \in \mathbb{N}$ s.t.

$$|f_n'(c) - g(c)| < \epsilon, \quad \forall n \geq N.$$

Now, take $K = \max\{H_1, N\} \in \mathbb{N}$. For $x \neq c$,

$$\left| \frac{f(x) - f(c)}{x-c} - g(c) \right|$$

Key $\longrightarrow$ $$\leq \left| \frac{f(x) - f(c)}{x-c} - \frac{f_K(x) - f_K(c)}{x-c} \right| + \left| \frac{f_K(x) - f_K(c)}{x-c} - f_K'(c) \right| + |f_K'(c) - g(c)|$$

$$< \frac{\epsilon}{2(b-a)} + \left| \frac{f_K(x) - f_K(c)}{x-c} - f_K'(c) \right| + \epsilon$$

Note: for the same $\epsilon > 0$, $\exists \delta = \delta(\epsilon, K, c) > 0$ s.t.

If $0 < |x-c| < \delta$, then $\left| \frac{f_K(x) - f_K(c)}{x-c} - f_K'(c) \right| < \epsilon$.

We then proved: $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. if $0 < |x-c| < \delta$, then

$$\left| \frac{f(x) - f(c)}{x-c} - g(c) \right| < \left(2 + \frac{1}{2(b-a)}\right)\epsilon$$

$\because \epsilon > 0$ is arbitrary, then $\lim_{x\to c} \frac{f(x) - f(c)}{x-c}$ exists and $= g(c)$

i.e. f is differentiable at c with $f'(c)=g(c)$

$\because c\in I$ is arbitrary

$\therefore$ f is differentiable on I with $f'=g$ on I. #

3rd issue: Interchange of Limit and Integral

Thm If $f_n \rightrightarrows f$ on $[a,b]$ with $f_n\in\mathcal{R}[a,b]$, $n=1,2,\cdots$, then $f\in\mathcal{R}[a,b]$ and
$$\lim_{n\to\infty}\int_a^b f_n = \int_a^b f$$

Pf $\boxed{\text{claim\#1}}$: $\left(\int_a^b f_n\right)$ is convergent in $\mathbb{R}$

so, $\exists A\in\mathbb{R}$ s.t. $\lim_{n\to\infty}\int_a^b f_n = A$

(use Cauchy Criterion)

In fact, $\because f_n\rightrightarrows f$ on $[a,b]$

$\therefore$ By Cauchy Criterion for unifor convergence

$\forall\epsilon>0$, $\exists H(\epsilon)\in\mathbb{N}$ s.t. if $m>n\geq H(\epsilon)$, then
$$\|f_m-f_n\|_{[a,b]}<\epsilon$$
i.e. $-\epsilon<f_m(x)-f_n(x)<\epsilon$, $\forall x\in[a,b]$

implying
$$-\epsilon(b-a)\leq\int_a^b f_m-\int_a^b f_n\leq\epsilon(b-a).$$
i.e.
$$\left|\int_a^b f_m-\int_a^b f_n\right|\leq\epsilon(b-a).$$
then, by Cauchy Criterion for convergence of sequence,

$\left(\int_a^b f_n\right)$ is convergent in $\mathbb{R}$

$\boxed{\text{claim\#2}}$: $f\in\mathcal{R}[a,b]$ with $\int_a^b f=A$ ($=\lim_{n\to\infty}\int_a^b f_n$ by claim#1)

$\boxed{\text{strategy}}$: $|S(f;\dot{\mathcal{P}})-A|\leq|S(f;\dot{\mathcal{P}})-S(f_n;\dot{\mathcal{P}})|$ ---- (small uniform in $\dot{\mathcal{P}}$)

$\left(\frac{\epsilon}{3}\right)$-strategy

$+\left|S(f_n;\dot{\mathcal{P}})-\int_a^b f_n\right|$ ---- (given n, small if $\|\dot{\mathcal{P}}\|$ small)

$+\left|\int_a^b f_n-A\right|$ --- (small if n large)

Proof of claim#2: Let $\epsilon > 0$.

Observe from the proof of claim#1: If $n \geq H(\epsilon)$, then

(*): $\|f_n - f\|_{[a,b]} \leq \epsilon$

(**): $\left|\int_a^b f_n - A\right| \leq \epsilon(b-a)$.

Choose $r \geq H(\epsilon)$ (e.g. $r = H(\epsilon) \in \mathbb{N}$),

$\forall \dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$, consider

$$
\begin{aligned}
|S(f;\dot{\mathcal{P}}) - A| &\leq |S(f;\dot{\mathcal{P}}) - S(f_r;\dot{\mathcal{P}})| \quad &\text{①}\\
&\quad + \left|S(f_r;\dot{\mathcal{P}}) - \int_a^b f_r\right| \quad &\text{②}\\
&\quad + \left|\int_a^b f_r - A\right| \quad &\text{③}
\end{aligned}
$$

For ①: $|S(f;\dot{\mathcal{P}}) - S(f_r;\dot{\mathcal{P}})|$

$$
\begin{aligned}
&= \left|\sum_{i=1}^n [f(t_i) - f_r(t_i)](x_i - x_{i-1})\right| \\
&\leq \sum_{i=1}^n |f_r(t_i) - f(t_i)|(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^n \|f_r - f\|(x_i - x_{i-1}) \\
&= \|f_r - f\| \sum_{i=1}^n (x_i - x_{i-1}) = \|f_r - f\|(b-a) \overset{(*)}{\leq} \epsilon(b-a)
\end{aligned}
$$

For ③: $\left|\int_a^b f_r - A\right| \leq \epsilon(b-a)$ by (**).

For ②: $\because f_r \in \mathcal{R}[a,b]$

$\therefore \exists \delta_{r,\epsilon} > 0$ s.t. $\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta_{r,\epsilon}$,

$$\left|S(f_r;\dot{\mathcal{P}}) - \int_a^b f_r\right| < \epsilon$$

Combining them, for any $\dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta_{r,\epsilon}$,

$$
\begin{aligned}
|S(f;\dot{\mathcal{P}}) - A| &\leq ① + ② + ③ \\
&\leq \epsilon(b-a) + \epsilon + \epsilon(b-a) \\
&= [1 + 2(b-a)]\epsilon.
\end{aligned}
$$

$\because \epsilon > 0$ is arbitrary

$\therefore f \in \mathcal{R}[a,b]$ with $\int_a^b f = A$. ##

Replace "uniform convergence" by $\begin{cases}\text{pointwise convergence} \\ \text{"uniform boundedness"} \\ + \\ \text{limit function} \in \mathcal{R}[a,b].\end{cases}$

Thm (Bounded Convergence Theorem)
Let $f_n \to f$ on $[a,b]$ with
- $f_n \in \mathcal{R}[a,b]$, $n \in \mathbb{N}$
- $f \in \mathcal{R}[a,b]$
- $\exists B > 0$ s.t. $\|f_n\|_{[a,b]} \le B$, $\forall n \in \mathbb{N}$

i.e. $|f_n(x)| \le B$, $\forall x \in [a,b]$, $\forall n \in \mathbb{N}$,

then $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f \left(= \int_a^b \lim_{n\to\infty} f_n \right)$. #

Pf: Omitted.

Thm (Dini's Theorem)
Let $f_n \to f$ on $[a,b]$ with
- (f_n) a monotone sequence of continuous function
- f continuous on $[a,b]$

then $f_n \rightrightarrows f$ on $[a,b]$

Pf: Assume: (f_n) is decreasing
Let $g_n = f_n - f$, $n = 1, 2, \cdots$.
then
- $g_1(x) \ge g_2(x) \ge \cdots \ge g_n(x) \ge g_{n+1}(x) \ge \cdots \ge 0$
- $g_n(x) \to 0$, $\forall x \in [a,b]$
- g_n is continuous on $[a,b]$.

Let $\epsilon > 0$.
Note: $\forall t \in [a,b]$, one can choose $n_{\epsilon,t} \in \mathbb{N}$ s.t.

$$0 \le g_{n_{\epsilon,t}}(t) < \epsilon/2,$$

$\because$ $g_{n_{\epsilon,t}}$ is continuous at t

$\therefore$ $\exists \delta_\epsilon(t) > 0$ s.t. $0 \le g_{n_{\epsilon,t}}(x) < \epsilon$, $\forall x \in (t-\delta_\epsilon(t), t+\delta_\epsilon(t)) \cap [a,b]$.

Observe (chap 5): $\exists N \in \mathbb{N}$ and $(t_i)_{i=1}^N$ s.t.

$$\underbrace{[a,b] \subset \bigcup_{i=1}^N I_i}_{\text{finite cover}}, \quad I_i \overset{\text{def.}}{=} (t_i - \delta_\epsilon(t_i), t_i + \delta_\epsilon(t_i)) \cap [a,b]$$

then $0 \le g_{n_{\epsilon, t_i}}(x) < \epsilon, \ \forall x \in I_i, \ \forall i \in \{1, \cdots, N\}$.

Define $M_\epsilon = \max\{n_{\epsilon,t_1}, n_{\epsilon,t_2}, \cdots, n_{\epsilon,t_N}\} \in \mathbb{N}$.

One can check: Let $m \ge M_\epsilon$, then

$\forall x \in [a,b], \ \exists i \in \{1, \cdots, N\}$ s.t. $x \in I_i$ and hence

$$0 \le g_m(x) \underset{\substack{\uparrow \\ \text{(monotone: decreasing)}}}{\le} g_{n_{\epsilon,t_i}}(x) < \epsilon$$

i.e. $\|g_m\|_{[a,b]} = \sup_{a \le x \le b} |g_m(x)| < \epsilon$,

We proved: $\forall \epsilon > 0, \ \exists M_\epsilon \in \mathbb{N}$ s.t. $\forall m \ge M_\epsilon$,

$$\|g_m\|_{[a,b]} < \epsilon.$$

i.e. $g_m \rightrightarrows 0$ on $[a,b]$

i.e. $f_n \rightrightarrows f$ on $[a,b]$. #