Topic #9 Pointwise and Uniform Convergence

Keywords:
* Sequence of functions
* pointwise convergence
* uniform convergence
* uniform norm
* Cauchy Criterion for uniform convergence

Notation:
Let $A \subseteq B$. Assume that for each $n \in \mathbb{N} = \{1, 2, \cdots\}$, there is a function

$$f_n : A \to \mathbb{R}.$$

We then say that (f_n) is a sequence of functions on A to $\mathbb{R}$.

RK: $\forall x \in A$, $(f_n(x))$ is a sequence of numbers in $\mathbb{R}$.

Def. (Pointwise Convergence)
Let
- (f_n) be a sequence of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$;
- $f : A_0 \to \mathbb{R}$ be a function with $A_0 \subseteq A$.

We say that the sequence (f_n) converges on A_0 to f if

$$\lim_{n\to\infty} f_n(x) = f(x), \quad \forall x \in A_0.$$

In this case,
- f is called the limit on A_0 of the sequence (f_n);
- (f_n) is said to be convergent on A_0, or (f_n) converges pointwise on A_0.

Rks: ① We usually take $A_0 = \{x \in A : (f_n(x)) \text{ is convergent in } \mathbb{R}\}$

② We write the poinwise convergence as:

$\begin{cases} \bullet\ f = \lim f_n \text{ on } A_0, \text{ or} \\ \bullet\ f_n \to f \text{ on } A_0 \end{cases}$ OR $\begin{cases} \bullet\ f(x) = \lim f_n(x) \text{ for } x \in A_0, \text{ or} \\ \bullet\ f_n(x) \to f(x) \text{ for } x \in A_0. \end{cases}$

(Slightly different notation with $f = \lim (f_n)$ on A_0 in the textbook page 242)

Examples:

(a) $\lim_{n\to\infty} \frac{x}{n} = x \cdot \lim_{n\to\infty} \frac{1}{n} = x \cdot 0 = 0$, for $x \in \mathbb{R}$. So

$$\lim_{n\to\infty} \frac{x}{n} = 0 \text{ for } x \in A_0 = \mathbb{R}$$

(b) Let $g_n(x) \overset{\text{def}}{=} x^n$, $x \in \mathbb{R}$, $n \in \mathbb{N}$.

If $|x| < 1$, then $x^n \to 0$

If $|x| > 1$, then x^n diverges

If $x = 1$, then $1^n = 1$ converges

If $x = -1$, then $(-1)^n$ diverges

$\therefore A_0 = \{x \in \mathbb{R} : (x^n) \text{ is convergent in } \mathbb{R}\}$
$= \{x \in \mathbb{R} : -1 < x \leq 1\}$

and

$$x^n \to g(x) \overset{\text{def.}}{=} \begin{cases} 0, & -1 < x < 1 \\ 1, & x = 1 \end{cases} \quad \text{on } (-1, 1]$$

note: g is discontinuous at 1.

(c) Let $h_n(x) \overset{\text{def}}{=} \frac{x^2 + nx}{n}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$.

Then, $\forall x \in \mathbb{R}$, $\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} \frac{x^2+nx}{n}$

$$= \lim_{n\to\infty} \left(\frac{x^2}{n} + x\right)$$

$$= x^2 \cdot \left(\lim_{n\to\infty} \frac{1}{n}\right) + x$$

$$= x^2 \cdot 0 + x$$

$$= x \overset{\text{def.}}{=} h(x)$$

$\therefore h_n(x) \to h(x)$ on $A_0 = \mathbb{R}$.

(d) Let $F_n(x) = \frac{1}{n} \sin(n(x+1))$, $x \in \mathbb{R}$, $n \in \mathbb{N}$.

Then, $\forall x \in \mathbb{R}$, $|F_n(x)| = \frac{1}{n} |\sin(n(x+1))| \leq \frac{1}{n} \to 0$ as $n \to \infty$

$\therefore F_n(x) \to F(x) \overset{\text{def.}}{=} 0$ on $A_0 = \mathbb{R}$.

Lemma (Pointwise Convergence)

A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ converges to a function $f: A_0 \to \mathbb{R}$ on $A_0 \subseteq A$ iff $\forall \epsilon > 0$ and $\forall x \in A_0$, $\exists K(\epsilon, x) \in \mathbb{N}$ s.t. if $n \geq K(\epsilon, x)$, then $|f_n(x) - f(x)| < \epsilon$.

Pf: just use $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in A_0$.

note: $K(\epsilon, x)$ may depend on both ϵ and x. #

e.g. ① Recall: $g_n(x) = x^n \to g(x) = 0$ on $(-1, 1) \overset{\text{def.}}{=} A_0$

For $|x| < 1$, consider

$$|g_n(x) - g(x)| = |x^n - 0| = |x|^n < \epsilon.$$

Let $0 < \epsilon < 1$, then $n \log|x| < \log \epsilon$, then $n > \dfrac{\log \frac{1}{\epsilon}}{\log \frac{1}{|x|}}$

Choose $K(\epsilon, x) = \left[\dfrac{\log \frac{1}{\epsilon}}{\log \frac{1}{|x|}} \right] + 1$

largest integer $\leq \dfrac{\log 1/\epsilon}{\log 1/|x|}$

Note: $K(\epsilon, x) \to \infty$ as $|x| \to 1$

so such $K(\epsilon, x)$ can NOT work for $x = 1$. #

② Recall: $F_n(x) \doteq \frac{1}{n} \sin(n(x+1)) \to F(x) = 0$ on $A_0 = \mathbb{R}$

$\forall x \in \mathbb{R}$, consider $|F_n(x) - F(x)| \leq \frac{1}{n} < \epsilon$ $(\Rightarrow n > \frac{1}{\epsilon})$

Then, choose $K(\epsilon) = [\frac{1}{\epsilon}] + 1$

note: such $K(\epsilon)$ is independent of x and works for all $x \in \mathbb{R}$.

#

Def. (Uniform Convergence)

A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ converges uniformly on $A_0 \subseteq A$ to a function $f: A_0 \to \mathbb{R}$ if $\forall \epsilon > 0$, $\exists K(\epsilon) \in \mathbb{N}$ (indep't of $x \in A_0$) s.t. if $n \geq K(\epsilon)$, then $|f_n(x) - f(x)| < \epsilon$, $\forall x \in A_0$.

In such case, we say:

Notation: "the sequence (f_n) is uniformly convergent on A_0",

and we write:

- $f_n \rightrightarrows f$ on A_0, or
- $f_n(x) \rightrightarrows f(x)$ for $x \in A_0$.

(In some other books, people also write: "$f_n \to f$ uniformly on A_0")

RKs ① Obvious to see: "uniform convergence $\Rightarrow$ pointwise convergence"

i.e. "$f_n \rightrightarrows f$ on A_0" $\Rightarrow$ "$f_n \to f$ on A_0"

② Negation gives "$f_n \not\rightrightarrows f$ on A_0":

Fact: A sequence (f_n) of functions $A \subseteq \mathbb{R}$ to $\mathbb{R}$ does NOT converge uniformly on $A_0 \subseteq A$ to a function $f: A_0 \to \mathbb{R}$

iff $\exists$ $\begin{cases} \bullet\ \epsilon_0 > 0 \\ \bullet \text{ a subsequence } (f_{n_k}) \text{ of } (f_n) \\ \bullet \text{ a sequence } (x_k) \text{ in } A_0 \end{cases}$

s.t.

$$|f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0, \quad \forall k \in \mathbb{N}.$$

Pf.: "$f_n \rightrightarrows f$ on A_0" $\Longleftrightarrow$ "$\forall \epsilon > 0$, $\exists K(\epsilon) \in \mathbb{N}$, s.t. $\forall n \geq K(\epsilon)$, $\forall x \in A_0$

$|f_n(x) - f(x)| < \epsilon$"

Negation:

"$f_n \not\rightrightarrows f$ on A_0" $\Longleftrightarrow$ "$\exists \epsilon_0 > 0$, $\forall l \in \mathbb{N}$, $\exists m_l \geq l$, $\exists y_l \in A_0$, s.t.

$|f_{m_l}(y_l) - f(y_l)| \geq \epsilon_0$.

then,

Proof of "only if" "$\Rightarrow$" need a careful choice of (n_k) and (x_k):

$k=1$: let $n_1 = m_1$, $x_1 = y_1$

$k=2$: Consider $l = m_1 + 1$, let $n_2 = m_{m_1+1} \geq n_1 + 1$,

$x_2 = y_{n_1+1}$

$k=3$: Consider $l = n_2 + 1$, let $n_3 = m_{n_2+1} \geq n_2 + 1$

$x_3 = y_{n_2+1}$

$\therefore$ repeated argument gives (n_k) and (x_k). ##

Examples

(a) Recall: $f_n(x) = \frac{x}{n} \to 0 = f(x)$ on $A_0 = \mathbb{R}$.

$\forall k \in \mathbb{N}$, choose $n_k = k$, $x_k = k$, then

$$\left| f_{n_k}(x_k) - f(x_k) \right| = \left| \frac{x_k}{n_k} - 0 \right| = \left| \frac{k}{k} - 0 \right| = 1.$$

Let $\epsilon_0 = 1$. Then, Fact gives: $f_n \not\rightrightarrows f$ on $\mathbb{R}$.

(b) Recall: $g_n(x) = x^n \to g(x) = \begin{cases} 0, & |x| < 1 \\ 1, & x = 1 \end{cases}$ on $A_0 = (-1, 1]$.

$\forall k \in \mathbb{N}$, choose $n_k = k$, $x_k = \left(\frac{1}{2}\right)^{1/k}$ $(|x_k| < 1)$

then

$$\left| g_{n_k}(x_k) - g(x_k) \right| = \left| \left[\left(\frac{1}{2}\right)^{1/k} \right]^k - 0 \right| = \frac{1}{2}.$$

Let $\epsilon_0 = \frac{1}{2}$. Then, Fact gives: $g_n \not\rightrightarrows g$ on $(-1, 1]$.

(c) Recall: $h_n(x) = \frac{x^2 + nx}{n} \to h(x) = x$ on $A_0 = \mathbb{R}$.

$\forall k \in \mathbb{N}$, choose $n_k = k$, $x_k = -k$,

then

$$\left| h_{n_k}(x_k) - h(x_k) \right| = \left| \frac{(x_k)^2 + n_k x_k}{n_k} - x_k \right|$$

$$= \left| \frac{(-k)^2 + k \cdot (-k)}{k} - (-k) \right|$$

$$= k \geq 1.$$

Let $\epsilon_0 = 1$, then Fact gives: $h_n \not\rightrightarrows h$ on $\mathbb{R}$. #

Def (uniform norm) (supnorm in some other books)

Let $\varphi : A \to \mathbb{R}$ be bounded, then

$$\|\varphi\|_A \overset{\text{def.}}{=} \sup\{ |\varphi(x)| : x \in A \}$$

exists and is called the uniform norm of φ on A.

RK: $\|\varphi\|_A \leq M \Leftrightarrow |\varphi(x)| \leq M, \ \forall x \in A$.

f, f_n may not be bounded,

e.g. $f(x) = \begin{cases} \frac{1}{x}, & 0 < x \le 1 \\ a \in \mathbb{R}, & x = 0 \end{cases}$ $\quad f_n(x) = f(x) + \frac{1}{n}, \ 0 \le x \le 1$

Lemma $f_n \rightrightarrows f$ on $A \iff \|f_n - f\|_A \to 0$

Pf: ($\Rightarrow$) Assume: $f_n \rightrightarrows f$ on A.

Let $\epsilon > 0$, then $\exists K(\epsilon) \in \mathbb{N}$ s.t. $\forall n \ge K(\epsilon)$, $\forall x \in A$,

$$|f_n(x) - f(x)| < \epsilon.$$

namely, $\forall n \ge K(\epsilon)$,

$$\|f_n - f\|_A = \sup_{x \in A} |f_n(x) - f(x)| \le \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this implies: $\|f_n - f\|_A \to 0$.

($\Leftarrow$) Assume: $\|f_n - f\|_A \to 0$.

Let $\epsilon > 0$, then $\exists H(\epsilon) \in \mathbb{N}$ s.t. $\forall n \ge H(\epsilon)$,

$$\|f_n - f\|_A = \sup_{x \in A} |f_n(x) - f(x)| < \epsilon$$

namely,

$$|f_n(x) - f(x)| < \epsilon, \ \forall x \in A.$$

$\therefore f_n \rightrightarrows f$ on A. #

Examples:

(a) $f_n(x) = \frac{x}{n}$, $x \in \mathbb{R}$;

$f(x) = 0$, $x \in \mathbb{R}$.

- $\|f_n - f\|_{\mathbb{R}} = \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = \sup_{x \in \mathbb{R}} \left|\frac{x}{n} - 0\right| = \sup_{x \in \mathbb{R}} \frac{|x|}{n}$

  does NOT exist.

- Consider $A = [0, 1]$, then

$$\|f_n - f\|_A = \sup_{0 \le x \le 1} |f_n(x) - f(x)|$$

$$= \sup_{0 \le x \le 1} \left|\frac{x}{n} - 0\right|$$

$$= \sup_{0 \le x \le 1} \frac{|x|}{n}$$

$$= \frac{1}{n} \to 0 \text{ as } n \to \infty$$

$\therefore$ By lemma above, $f_n \rightrightarrows f$ on $A = [0, 1]$.

(b) Recall: $g_n(x) \overset{\text{def.}}{=} x^n \longrightarrow g(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$ on $A = [0,1]$.

Consider

$$\|g_n - g\|_A = \sup_{0 \le x \le 1} |g_n(x) - g(x)|$$

$$= \sup_{0 \le x \le 1} |x^n - g(x)|, \text{ where } x^n - g(x) = \begin{cases} x^n & 0 \le x < 1 \\ 0 & x = 1 \end{cases}$$

$$= 1. \quad \left(\because \sup_{0 \le x < 1} x^n = 1 \text{ as } x^n \underset{(x \to 1-)}{\to} 1 \right)$$

$\therefore \|g_n - g\|_A \not\to 0$, then $g_n \not\rightrightarrows g$ on $A = [0,1]$.

(c) Recall: $h_n(x) \overset{\text{def.}}{=} \frac{x^2 + nx}{n} \to h(x) = x$ on $\mathbb{R}$

Note: $h_n(x) - h(x) = \frac{x^2}{n}$ is unbounded on $\mathbb{R}$

$\therefore \|h_n - h\|_{\mathbb{R}}$ is NOT well-defined.

Let $A = [0, 8]$ (generally a bounded subset of $\mathbb{R}$)

Compute $\|h_n - h\|_A = \sup_{0 \le x \le 8} |h_n(x) - h(x)|$

$$= \sup_{0 \le x \le 8} \frac{x^2}{n}$$

$$= \frac{64}{n} \longrightarrow 0$$

$\therefore h_n \rightrightarrows h$ on $A = [0, 8]$.

(d) Recall: $F_n(x) \overset{\text{def.}}{=} \frac{1}{n} \sin(n(x+1)) \longrightarrow F(x) = 0$ on $\mathbb{R}$

Compute $\|F_n - F\|_{\mathbb{R}} = \sup_{x \in \mathbb{R}} |F_n(x) - F(x)|$

$$= \sup_{x \in \mathbb{R}} \frac{|\sin(n(x+1))|}{n}$$

$$\le \frac{1}{n} \longrightarrow 0 \qquad \left(\text{Ex: Show } \|F_n - F\|_{\mathbb{R}} = \tfrac{1}{n} \right)$$

$\therefore F_n \rightrightarrows F$ on $\mathbb{R}$.

(e) $G_n(x) \overset{\text{def.}}{=} x^n(1-x) \longrightarrow G(x)=0$ on $A=[0,1]$ (Exercise).

To consider the uniform convergence,

note: $\forall x \in [0,1]$, $|G_n(x) - G(x)| = G_n(x) \geq 0$:

- $G_n(0) = 0 = G_n(1)$
- Solve $0 = G_n'(x)$

$$= [x^n(1-x)]'$$
$$= nx^{n-1}(1-x) - x^n$$
$$= x^{n-1}[n-(n+1)x]$$

$\therefore x = \frac{n}{n+1}$ is the only interior critical p't

then $G_n(x) \geq 0$ attains its maximum at $x = \frac{n}{n+1}$

thus $\|G_n - G\|_{[0,1]} = \sup_{0 \leq x \leq 1} G_n(x)$

$$= G_n\left(\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n\left(1 - \frac{n}{n+1}\right)$$
$$= \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{n+1}$$

where $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, $\lim_{n\to\infty}\frac{1}{n+1} = 0$.

$\therefore \|G_n - G\|_{[0,1]} \to 0$

$\therefore G_n \rightrightarrows G$ on $[0,1]$. #

Thm (Cauchy Criterion for Uniform Convergence)

(f_n) is uniformly convergent on A

iff $\forall \epsilon > 0$, $\exists H(\epsilon) \in \mathbb{N}$ s.t. $\forall m, n \geq H(\epsilon)$,

$$\|f_m - f_n\|_A < \epsilon.$$

Pf ($\Rightarrow$) Assume: (f_n) is uniformly convergent on A,

then $\exists f: A \to \mathbb{R}$ s.t. $f_n \rightrightarrows f$ on A

Let $\epsilon > 0$.

$\because \|f_n - f\|_A \to 0$

$\therefore \exists K(\frac{\epsilon}{2}) \in \mathbb{N}$ s.t. $\forall n \geq K(\frac{\epsilon}{2})$, $\|f_n - f\|_A < \frac{\epsilon}{2}$.

therefore, $\forall m, n \geq K(\frac{\epsilon}{2}) \overset{\text{def.}}{=} H(\epsilon)$

$$\|f_m - f_n\|_A = \sup_{x \in A} |f_m(x) - f_n(x)|$$

$$\leq \sup_{x\in A} \left[|f_m(x) - f(x)| + |f_n(x) - f(x)| \right]$$

$$\leq \sup_{x\in A} |f_m(x) - f(x)| + \sup_{x\in A} |f_n(x) - f(x)|$$

$$= \|f_m - f\|_A + \|f_n - f\|_A$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \#$$

($\Leftarrow$) Assume: $\forall \epsilon > 0$, $\exists H(\epsilon) \in \mathbb{N}$, s.t. $\forall m, n \geq H(\epsilon)$,

$$\|f_m - f_n\|_A < \epsilon.$$

this tells:

$\forall \epsilon > 0$, $\exists H(\epsilon) \in \mathbb{N}$, s.t. $\forall m, n \geq H(\epsilon)$, $\forall x \in A$,

(*) $|f_m(x) - f_n(x)| \leq \|f_m - f_n\|_A < \epsilon$.

Then, $\forall x \in A$, by Cauchy Criterion for Convergence of sequence of numbers in $\mathbb{R}$, $(f_n(x))$ is convergent.

Define

$$f(x) = \lim_{n\to\infty} f_n(x), \quad x \in A.$$

We claim: $f_n \rightrightarrows f$ on A.

In fact, let $m \to \infty$ in (*), we have:

$\forall \epsilon > 0$, $\exists H(\epsilon) \in \mathbb{N}$, s.t. $\forall n \geq H(\epsilon)$,

$|f(x) - f_n(x)| \leq \epsilon$, $\forall x \in A$

namely, $\|f_n - f\|_A \leq \epsilon$.

$\therefore \lim_{n\to\infty} \|f_n - f\|_A = 0$

i.e. (f_n) is uniformly convergent on A with $f_n \rightrightarrows f$ on A. ##

RK: Slight modification of statement, show that:

Let (f_n) be a sequence of functions that are bounded on A.
Then, $\exists$ a function f bounded on A s.t. $f_n \rightrightarrows f$ on A
iff $\forall \epsilon > 0$, $\exists H(\epsilon) \in \mathbb{N}$ s.t. $\forall m, n \geq H(\epsilon)$,
$\|f_m - f_n\|_A < \epsilon$.

pf: ($\Rightarrow$) the same as before

($\Leftarrow$) As before, $\exists f: A \to \mathbb{R}$ s.t. $f_n \rightrightarrows f$ on A.

It remains to show: f is bounded on A.

In fact,
$$\|f\|_A = \sup_{x\in A} |f(x)|$$
$$\le \sup_{x\in A} \left[|f(x) - f_n(x)| + |f_n(x)| \right]$$
$$\le \sup_{x\in A} |f_n(x) - f(x)| + \sup_{x\in A} |f_n(x)|$$
$$= \|f_n - f\|_A + \|f_n\|_A, \quad \forall n \in \mathbb{N}.$$

$\because \|f_n - f\|_A \to 0$

$\therefore \exists N \in \mathbb{N}$ s.t. $\|f_n - f\|_A \le 1, \forall n \ge N$.

In particular, $\|f_N - f\|_A \le 1$.

therefore
$$\|f\|_A \le \|f_N - f\|_A + \|f_N\|_A \le 1 + \|f_N\|_A.$$

$\because$ f_n is bouned on A for each $n \in \mathbb{N}$

$\therefore \|f_N\|_A = \sup_{x\in A} |f_N(x)|$ is finite.

$\therefore \|f\|_A \le 1 + \|f_N\|_A < \infty$

i.e. f is bounded on A. #

Note: ① "uniform limit" of bounded functions is still bounded.

② (Exercise) $\boxed{\begin{array}{l}\text{If } (f_n) \text{ is uniformly convergent on } A \\ \text{with each } f_n \text{ bounded on } A, \\ \text{then } (f_n) \text{ is also uniformly bounded on } A, \\ \text{i.e. } \exists M \in \mathbb{R} \text{ s.t. } \|f_n\|_A \le M < \infty, \\ \text{i.e. } |f_n(x)| \le M, \forall x \in A. \end{array}}$ #