Topic #8 Darboux Integral

Keywords:
- * Upper and lower integrals
- * Darboux integrable functions
- * Cauchy-like integrability criterion for Darboux integrals
- * Equivalence of Riemann and Darboux integrability

Def. (Upper and Lower Sums)

Let
- $f: [a,b] \to \mathbb{R}$ bounded
- $P = (x_0, x_1, \cdots, x_n) = \{[x_{k-1}, x_k]\}_{k=1}^{n}$: partition of $[a,b]$
- $m_k = \inf_{x \in [x_{k-1}, x_k]} f(x)$, $M_k = \sup_{x \in [x_{k-1}, x_k]} f(x)$, $k = 1, \cdots, n$

(exist & finite)

then

$$L(f;P) \overset{\text{def.}}{=} \sum_{k=1}^{n} m_k (x_k - x_{k-1}) \text{ : the lower sum of } f \text{ corresponding to } P$$

$$U(f;P) \overset{\text{def.}}{=} \sum_{k=1}^{n} M_k (x_k - x_{k-1}) \text{ : the upper sum of } f \text{ corresponding to } P$$

RKS: ① If $f \geq 0$, then

$L(f;P)$: area of union of rectangles with base $[x_{k-1}, x_k]$ and height m_k

$U(f;P)$: area of union of rectangles with base $[x_{k-1}, x_k]$ and height M_k.

② m_k, M_k may not be attained at points in $[x_{k-1}, x_k]$, then $L(f;P)$ and $U(f;P)$ may NOT be the Riemann sums.

Fact#1: $L(f;P) \leq U(f;P)$, $\forall$ partition P of $[a,b]$.

Pf: $m_k = \inf_{[x_{k-1}, x_k]} f \leq \sup_{[x_{k-1}, x_k]} f = M_k$

$$\therefore L(f;P) = \sum_{k=1}^{n} m_k \underbrace{(x_k - x_{k-1})}_{>0} \leq \sum_{k=1}^{n} M_k (x_k - x_{k-1}) = U(f;P).$$

#

Def. Let $\mathcal{P}$ and Q be partitions of $[a,b]$.
We say that Q is a refinement of $\mathcal{P}$ if $\mathcal{P} \subseteq Q$, namely, each partition point of $\mathcal{P}$ belongs to Q. #

Rk. Let $\mathcal{P} = (x_0, x_1, \cdots, x_n)$
$Q = (y_0, y_1, \cdots, y_n)$.
If Q is a refinement of $\mathcal{P}$, then $x_k \in Q$, $\forall k \in \{0, \cdots, n\}$.
In such case, each subinterval $[x_{k-1}, x_k]$ of $\mathcal{P}$ can be further divided into subintervals of Q:

$$[x_{k-1}, x_k] = [\underset{x_{k-1}}{\overset{\shortparallel}{y_{j-1}}}, y_j] \cup [y_j, y_{j+1}] \cup \cdots \cup [y_{h-1}, \underset{x_k}{\overset{\shortparallel}{y_h}}]$$ #

Fact#2. Let $f: [a,b] \to \mathbb{R}$ be bounded,
then $L(f;\mathcal{P}) \leq L(f;Q)$ and $U(f;Q) \leq U(f;\mathcal{P})$,
$\forall$ partitions $\mathcal{P}$ & Q of $[a,b]$ with Q being a refinement of $\mathcal{P}$.

Pf: Let $\mathcal{P} = (x_0, x_1, \cdots, x_n)$ be a partition of $[a,b]$.
Simple case of Q:
$Q = (x_0, x_1, \cdots, x_{k-1}, z, x_k, \cdots, x_n)$
is a refinement Q by adjoining one point z into $\mathcal{P}$.
Then,

$$m_k' \overset{\text{def.}}{=} \inf_{[x_{k-1}, z]} f \geq \inf_{[x_{k-1}, x_k]} f = m_k$$

$$m_k'' \overset{\text{def.}}{=} \inf_{[z, x_k]} f \geq \inf_{[x_{k-1}, x_k]} f = m_k$$

Therefore,

$$L(f;Q) = \sum_{i=1}^{k-1} m_i(x_i - x_{i-1}) + m_k'(z - x_{k-1}) + m_k''(x_k - z) + \sum_{i=k+1}^{n} m_i(x_i - x_{i-1})$$

$$\geq \sum_{\substack{i=1 \\ i \neq k}}^{n} m_i(x_i - x_{i-1}) + m_k(z - x_{k-1}) + m_k(x_k - z)$$

$$= \sum_{i \neq k} m_i(x_i - x_{i-1}) + m_k(x_k - x_{k-1}) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

$$= L(f;\mathcal{P}).$$

Similarly, $U(f;Q) \leq U(f;P)$. (Exercise).

General case of Q:

Q is a refinement of P

then Q can be obtained from P by adjoining a finite number of points to P one at a time.

then repeating the argument in the special case gives

$$L(f;P) \leq L(f;Q) \quad \text{and} \quad U(f;P) \geq U(f;Q).$$

#

Fact #3: Let $f:[a,b] \to \mathbb{R}$ be bounded,
then $L(f;P_1) \leq U(f;P_2)$, $\forall$ partitions P_1, P_2 of $[a,b]$.

Pf: Let $Q = P_1 \cup P_2$, then Q is a refinement of P_1 and P_2, so

$$\underset{\text{Fact \#2}}{L(f;P_1) \leq} L(f;Q) \underset{\text{Fact \#1}}{\leq} U(f;Q) \underset{\text{Fact \#2}}{\leq} U(f;P_2).$$

#

Def. $\mathcal{P}([a,b]) \overset{\text{def.}}{=}$ the collection of all partitions of $[a,b]$.

Let $f:[a,b] \to \mathbb{R}$ be bounded,

then

$L(f) \overset{\text{def.}}{=} \sup\{L(f;P) : P \in \mathcal{P}([a,b])\}$

called the lower integral of f on $[a,b]$;

$U(f) \overset{\text{def.}}{=} \inf\{U(f;P) : P \in \mathcal{P}([a,b])\}$

called the upper integral of f on $[a,b]$.

Thm: Let $f:[a,b] \to \mathbb{R}$ be bounded,
then $L(f)$ and $U(f)$ exist with $L(f) \leq U(f)$.

Pf: ① $L(f), U(f) \in \mathbb{R}$: In fact,

$\because$ f is bounded on $I = [a,b]$

$\therefore$ $m_I \overset{\text{def.}}{=} \inf\limits_{[a,b]} f$, $M_I \overset{\text{def.}}{=} \sup\limits_{[a,b]} f$, both exist.

Therefore, $\forall\, P \in \mathcal{P}([a,b])$, it holds:

$$m_I(b-a) \le L(f;P) \le U(f;P) \le M_I(b-a).$$

$$\therefore\ \left.\begin{aligned} L(f) &= \sup_{P\in\mathcal{P}([a,b])} L(f;P) \\ U(f) &= \inf_{P\in\mathcal{P}([a,b])} U(f;P) \end{aligned}\right\} \text{ both exist, with } \begin{aligned} L(f) &\ge m_I(b-a) \\ U(f) &\le M_I(b-a). \end{aligned}$$

② $L(f) \le U(f)$. In fact, it holds that

$$L(f;P_1) \le U(f;P_2),\ \forall\, P_1, P_2 \in \mathcal{P}([a,b]).$$

Fix $P_2 \in \mathcal{P}([a,b])$ and take the supremum for $P_1 \in \mathcal{P}([a,b])$, then

$$L(f) = \sup_{P_1\in\mathcal{P}([a,b])} L(f;P_1) \le U(f;P_2),$$

$$\forall\, P_2 \in \mathcal{P}([a,b]).$$

Further take the infimum for $P_2 \in \mathcal{P}([a,b])$, then

$$L(f) \le \inf_{P_2\in\mathcal{P}([a,b])} U(f;P_2) = U(f).$$

#

Def. Let $f:[a,b]\to\mathbb{R}$ be bounded, then f is said to be Darboux integrable on $[a,b]$ if $L(f)=U(f)$. In this case, the Darboux integral of f over $[a,b]$ is defined to be the value $L(f)=U(f)\in\mathbb{R}$. #

RK: We use the same notation $\int_a^b f$ or $\int_a^b f(x)\,dx$ for the Darboux integral because we will establish the equivalence of the Darboux and Riemann integrals. #

Examples:

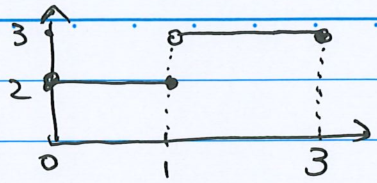
(a) A constant function is Darboux integrable.

In fact, let $f\equiv c$ on $[a,b]$. For any partition P of $[a,b]$,

$$L(f;P) = c(b-a) = U(f;P) \quad (\text{Exercise})$$

$$\therefore\ L(f) = c(b-a) = U(f).\ \#$$

(b) Revisit $g(x) = \begin{cases} 3, & 1 < x \le 3 \\ 2, & 0 \le x \le 1 \end{cases}$

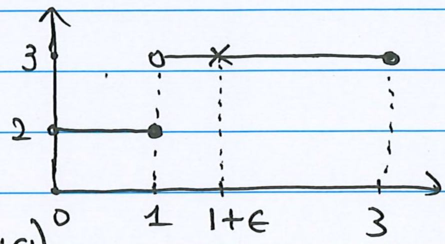


(Recall: g is Riemann integrable with $\int_0^3 g = 8$)

g is bounded on $[0,3]$.

Let $\epsilon > 0$, consider the partition:

$$P_\epsilon = (0, 1, 1+\epsilon, 3)$$



Compute:

$$U(g; P_\epsilon) = 2\cdot(1-0) + 3\cdot(1+\epsilon-1) + 3\cdot(3-(1+\epsilon))$$

$$= 2 + 3\epsilon + 6 - 3\epsilon = 8$$

$$L(g; P_\epsilon) = 2\cdot(1-0) + 2\cdot(1+\epsilon-1) + 3\cdot(3-(1+\epsilon))$$

$$\inf_{1 \le x \le 1+\epsilon} g(x) = 2$$

$$= 2 + 2\epsilon + 6 - 3\epsilon = 8 - \epsilon.$$

therefore

$$U(g) = \inf_{P \in \mathcal{P}([0,3])} U(g;P) \le U(g; P_\epsilon) = 8$$

$$L(g) = \sup_{P \in \mathcal{P}([0,3])} L(g;P) \ge L(g; P_\epsilon) = 8 - \epsilon$$

Therefore,

$$8 - \epsilon \le L(g) \le U(g) \le 8,$$

$\because$ $\epsilon > 0$ can be arbitrary

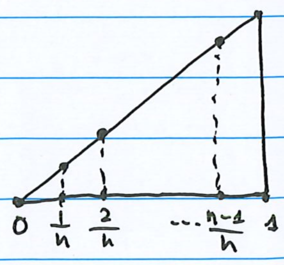
$\therefore$ $L(g) = U(g) = 8$.

then g is Darboux integrable with $\int_0^3 g = 8$. #

(c) Revisit $h(x) = x$ on $[0,1]$.

h is bounded on $[0,1]$.

Consider a uniform partition $P_n = (0, \frac{1}{n}, \frac{2}{n}, \cdots, 1)$ of $[0,1]$



Compute:

$$U(h; P_n) = \frac{1}{n}\left(\frac{1}{n}-0\right) + \frac{2}{n}\left(\frac{2}{n}-\frac{1}{n}\right) + \cdots + 1\cdot\left(1 - \frac{n-1}{n}\right)$$

$$= \frac{1}{n^2}(1+2+\cdots+n) = \frac{n(n+1)}{2n^2} = \frac{1}{2}\left(1+\frac{1}{n}\right).$$

$$L(h;P_n) = 0\cdot\frac{1}{n} + \frac{1}{n}\cdot\frac{1}{n} + \cdots + \frac{n-1}{n}\cdot\frac{1}{n}$$
$$= \frac{1}{n^2}\left[1+2+\cdots+(n-1)\right] = \frac{n(n-1)}{2n^2} = \frac{1}{2}\left(1-\frac{1}{n}\right).$$

This implies:
$$\frac{1}{2}\left(1-\frac{1}{n}\right) \le L(h) \le U(h) \le \frac{1}{2}\left(1+\frac{1}{n}\right).$$

Letting $n\to\infty$, it holds: $L(h) = U(h) = \frac{1}{2}$.

$\therefore$ $h(x) = x$ is Darboux integrable on $[0,1]$ with $\int_0^1 h = \frac{1}{2}$. #

Sum of such strategy:

Find a sequence of partitions (P_n) of $[a,b]$.

We have: $L(f;P_n) \le L(f) \le U(f) \le U(f;P_n)$

If $\exists L \in \mathbb{R}$ s.t. $L = \lim_{n\to\infty} L(f;P_n) = \lim_{n\to\infty} U(f;P_n)$

then f is Darboux integrable with $L = \int_a^b f$.

(d) Revisit the Dirichlet function

$$f(x) = \begin{cases} 1, & x \text{ rational in } [0,1] \\ 0, & x \text{ irrational in } [0,1]. \end{cases}$$

(Recall: f is NOT Riemann integrable)

f is bounded with $0 \le f(x) \le 1$, $\forall x \in [0,1]$.

Let $P = (x_0, x_1, \cdots, x_n)$ be a partition of $[0,1]$,

then $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} = f(t_k) = 0$

($\exists t_k \in [x_{k-1}, x_k]$ is irrational)

$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} = f(r_k) = 1$

($\exists r_k \in [x_{k-1}, x_k]$ is rational)

$\therefore$ $L(f;P) = \sum_k m_k (x_k - x_{k-1}) = 0$

$U(f;P) = \sum_k M_k (x_k - x_{k-1}) = \sum_k (x_k - x_{k-1}) = 1$

Thus

$$L(f) = \sup_{P \in \mathcal{P}([0,1])} L(f;P) = 0, \quad U(f) = \inf_{P\in\mathcal{P}([0,1])} U(f;P) = 1$$

$\therefore$ $L(f) = 0 < 1 = U(f)$

then f is NOT Darboux integrable. #

Thm (Integrability Criterion)

Let $f:[a,b]\to\mathbb{R}$ be bounded, then

f is Darboux integrable on $[a,b]$ iff $\forall \epsilon>0$, $\exists$ a partition P_ϵ of $[a,b]$ s.t. $U(f;P_\epsilon)-L(f;P_\epsilon)<\epsilon$.

pf: "$\Rightarrow$" Let f be Darboux integrable on $[a,b]$,

then $L(f)=U(f)\in\mathbb{R}$.

Let $\epsilon>0$.

$\because$ $L(f)=\sup\limits_{P\in\mathcal{P}([a,b])} L(f;P)$

$\therefore$ $\exists$ a partition P_1 of $[a,b]$ s.t.

$$L(f)-\frac{\epsilon}{2}<L(f;P_1)$$

Similarly,

$\because$ $U(f)=\inf\limits_{P\in\mathcal{P}([a,b])} U(f;P)$

$\therefore$ $\exists$ a partition P_2 of $[a,b]$ s.t.

$$U(f;P_2)<U(f)+\frac{\epsilon}{2}.$$

Define $P_\epsilon=P_1\cup P_2$,

then P_ϵ is a refinement of P_1 and P_2

$$\begin{aligned}\therefore\ L(f)-\frac{\epsilon}{2} &< L(f;P_1)\\ &\leq L(f;P_\epsilon)\\ &\leq U(f;P_\epsilon)\\ &\leq U(f;P_2)\\ &< U(f)+\frac{\epsilon}{2}\end{aligned}$$

then $U(f;P_\epsilon)-L(f;P_\epsilon)<\left[U(f)+\frac{\epsilon}{2}\right]-\left[L(f)+\frac{\epsilon}{2}\right]=\epsilon$

($\because L(f)=U(f)$)

"$\Leftarrow$" Let $\epsilon>0$. By the assumption, $\exists P_\epsilon$ of $[a,b]$ s.t. $U(f;P_\epsilon)-L(f;P_\epsilon)<\epsilon$.

Then,

$$0\leq U(f)-L(f)\leq U(f;P_\epsilon)-L(f;P_\epsilon)<\epsilon.$$

$\because$ $\epsilon>0$ is arbitrary

$\therefore$ $U(f)-L(f)=0$, then f is Darboux integrable on $[a,b]$. #

Coro: Let $f:[a,b]\to\mathbb{R}$ be bounded.

If there is a sequence of partition P_n, $n=1,2,\cdots$, of $[a,b]$ s.t.

$$\lim_{n\to\infty}\left[U(f;P_n)-L(f;P_n)\right]=0$$

then f is Darboux integrable on $[a,b]$ with

$$\int_a^b f=\lim_{n\to\infty}L(f;P_n)=\lim_{n\to\infty}U(f;P_n).$$

Pf: The condition implies: $\exists L\in\mathbb{R}$ s.t.

$$\lim_{n\to\infty}L(f;P_n)=L=\lim_{n\to\infty}U(f;P_n).$$

Note (as before):

$$L(f;P_n)\le L(f)\le U(f)\le U(f;P_n),\quad \forall n\ge 1.$$

Then, letting $n\to\infty$ and using the squeeze law,

$$L(f)=U(f)=L=\lim_{n\to\infty}L(f;P_n)=\lim_{n\to\infty}U(f;P_n).$$

$\therefore$ f is Darboux integrable on $[a,b]$

with $\quad L(f)=U(f)=\int_a^b f=L$. #

Thm. Let $f:[a,b]\to\mathbb{R}$ be either continuous or monotone, then f is Darboux integrable on $[a,b]$.

Pf: Case $f:[a,b]\to\mathbb{R}$ is continuous:

Let $P=(x_0,x_1,\cdots,x_n)$ be a partition of $[a,b]$.

$\because$ f is continuous on $[a,b]$

$$\therefore\ M_k=\sup_{[x_{k-1},x_k]}f=f(v_k)\quad\text{for some } v_k\in[x_{k-1},x_k]$$

$$m_k=\inf_{[x_{k-1},x_k]}f=f(u_k)\quad\text{for some } u_k\in[x_{k-1},x_k].$$

then $L(f;P)=\sum_k m_k(x_k-x_{k-1})=\sum_k f(u_k)(x_k-x_{k-1})=\int_a^b \alpha$

where $\alpha(x)=\begin{cases} f(u_k), & x\in[x_{k-1},x_k),\ k=1,\cdots,n-1\\ f(u_n) & x\in[x_{n-1},x_n].\end{cases}$

is a step function

$$U(f;P) = \sum_k M_k (x_k - x_{k-1}) = \sum_k f(v_k)(x_k - x_{k-1}) = \int_a^b \omega$$

where $\omega(x) = \begin{cases} f(v_k), & x \in [x_{k-1}, x_k),\ k=1,\cdots,n-1 \\ f(v_n) & x \in [x_{n-1}, x_n], \end{cases}$

is a step function.

As in the proof of Thm 7.2.7 (textbook, page 212)

$\forall \epsilon > 0$, $\exists$ a partition P_ϵ of $[a,b]$

and step functions $\alpha_\epsilon(x)$, $\omega_\epsilon(x)$

s.t. $0 \le \int_a^b (\omega_\epsilon - \alpha_\epsilon) < \epsilon$.

therefore,

$$U(f;P_\epsilon) - L(f;P_\epsilon) = \left(\int_a^b \omega_\epsilon\right) - \left(\int_a^b \alpha_\epsilon\right)$$

$$= \int_a^b (\omega_\epsilon - \alpha_\epsilon)$$

$$< \epsilon.$$

By Integrability Criterion, f is Darboux integrable on $[a,b]$.

Case: $f : [a,b] \to \mathbb{R}$ is monotone.

for instance, assume: increasing.

Let $P = (x_0, x_1, \cdots, x_n)$ be a partition of $[a,b]$.

then $M_k = \sup_{[x_{k-1},x_k]} f = f(x_k)$, $m_k = \inf_{[x_{k-1},x_k]} f = f(x_{k-1})$.

$$\therefore\ L(f;P) = \sum_k m_k (x_k - x_{k-1}) = \sum_k f(x_{k-1})(x_k - x_{k-1}) = \int_a^b \alpha$$

where $\alpha(x) = \begin{cases} f(x_{k-1}), & x \in [x_{k-1}, x_k),\ k=1,\cdots,n-1 \\ f(x_{n-1}), & x \in [x_{n-1}, x_n]. \end{cases}$

is a step function.

$$U(f;P) = \sum_k M_k (x_k - x_{k-1}) = \sum_k f(x_k)(x_k - x_{k-1}) = \int_a^b \omega$$

where $\omega(x) = \begin{cases} f(x_k), & x \in [x_{k-1}, x_k),\ k = 1, 2, \cdots, n-1 \\ f(x_n), & x \in [x_{n-1}, x_n] \end{cases}$

is a step function.

Recall: As in the pf of Thm 7.2.8 (Textbook, page 212)

consider uniform partitions P_n of $[a,b]$ with $\|P_n\| = \frac{b-a}{n}$

step functions $\alpha_n(x)$, $\omega_n(x)$ satisfy: $\int_a^b (\omega_n - \alpha_n) = \frac{b-a}{n}[f(b) - f(a)]$

Thus,

$$0 \le U(f;P_n) - L(f;P_n)$$
$$= \left(\int_a^b w_n\right) - \left(\int_a^b \alpha_n\right)$$
$$= \int_a^b (w_n - \alpha_n) = \frac{b-a}{n}\,[f(b)-f(a)] \longrightarrow 0 \text{ as } n \to \infty.$$

By the Corollary of Integrability Criterion,

f is Darboux integrable on $[a,b]$. ##

Thm (Equivalence Theorem)

$f:[a,b]\to\mathbb{R}$ is Darboux integrable iff it is Riemann integrable.

In this case, the integrals in sense of Darboux and Riemann are equal to each other.

Pf: "$\Rightarrow$" Let $\epsilon > 0$.

$\because$ f is Darboux integrable on $[a,b]$

$\therefore$ By Integrability Criterion,

$\exists$ partition P_ϵ of $[a,b]$ s.t.

$$U(f;P_\epsilon) - L(f;P_\epsilon) < \epsilon.$$

Set $P_\epsilon = (x_0, x_1, \cdots, x_n)$

Define step functions ($\therefore$ Riemann integrable)

$$\alpha_\epsilon(x) = m_k = \inf_{[x_{k-1},x_k]} f, \quad \forall x \in [x_{k-1}, x_k) \text{ if } k=1,\cdots,n-1, \quad [x_{n-1},x_n] \text{ if } k=n$$

and

$$w_\epsilon(x) = M_k = \sup_{[x_{k-1},x_k]} f, \quad \forall x \in [x_{k-1}, x_k) \text{ if } k=1,\cdots,n-1, \quad [x_{n-1},x_n] \text{ if } k=n$$

then $\alpha_\epsilon(x) \le f(x) \le w_\epsilon(x)$, $\forall x \in [a,b]$.

and

$$\int_a^b \alpha_\epsilon = \sum_k m_k (x_k - x_{k-1}) = L(f;P_\epsilon)$$
$$\int_a^b w_\epsilon = \sum_k M_k (x_k - x_{k-1}) = U(f;P_\epsilon).$$

$$\therefore \int_a^b (w_\epsilon - \alpha_\epsilon) = \left(\int_a^b w_\epsilon\right) - \left(\int_a^b \alpha_\epsilon\right) = U(f;P_\epsilon) - L(f;P_\epsilon) < \epsilon.$$

$\therefore$ Squeeze Theorem (Thm 7.2.3, page 209, textbook) gives that

f is Riemann integrable. #

"$\Leftarrow$" Let $\epsilon > 0$.

$\because f \in \mathcal{R}[a,b]$

$\therefore \exists \delta_\epsilon > 0$ s.t. $\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta_\epsilon$,

$$|S(f;\dot{\mathcal{P}}) - A| < \epsilon, \text{ where } A = \int_a^b f.$$

Let $\mathcal{P} = (x_0, x_1, \cdots, x_n)$ be a partition of $[a,b]$ with $\|\mathcal{P}\| < \delta_\epsilon$.

Note: f is bounded on $[a,b]$.

then $M_k = \sup_{[x_{k-1},x_k]} f$, $m_k = \inf_{[x_{k-1},x_k]} f$ exist

and $\exists t_k \in [x_{k-1}, x_k]$ and $t_k' \in [x_{k-1}, x_k]$ s.t.

$$f(t_k) > M_k - \frac{\epsilon}{b-a}$$

$$f(t_k') < m_k + \frac{\epsilon}{b-a}.$$

① Consider the tagged partition $\dot{\mathcal{P}} = \{[x_{k-1}, x_k], t_k\}_{k=1}^n$ and compute

$$\begin{aligned} S(f;\dot{\mathcal{P}}) &= \sum_{k=1}^n f(t_k)(x_k - x_{k-1}) \\ &> \sum_{k=1}^n \left(M_k - \frac{\epsilon}{b-a}\right)(x_k - x_{k-1}) \\ &= \sum_{k=1}^n M_k(x_k - x_{k-1}) - \frac{\epsilon}{b-a}\sum_{k=1}^n (x_k - x_{k-1}) \\ &= U(f;\mathcal{P}) - \epsilon \end{aligned}$$

use $|S(f;\dot{\mathcal{P}}) - A| < \epsilon$, obtain

$$U(f;\mathcal{P}) < S(f;\dot{\mathcal{P}}) + \epsilon < A + 2\epsilon$$

$$\therefore U(f) = \inf_{P \in \mathcal{P}([a,b])} U(f;P) < A + 2\epsilon$$

$\because \epsilon > 0$ is arbitrary

$\therefore U(f) \le A$.

② Similarly, consider the tagged partition $\dot{\mathcal{P}}' = \{[x_{k-1}, x_k], t_k'\}_{k=1}^n$ and compute

$$\begin{aligned} S(f;\dot{\mathcal{P}}') &= \sum_{k=1}^n f(t_k')(x_k - x_{k-1}) \\ &< \sum_{k=1}^n \left(m_k + \frac{\epsilon}{b-a}\right)(x_k - x_{k-1}) = L(f;\mathcal{P}') + \epsilon \end{aligned}$$

use $|S(f;\dot{\mathcal{P}}') - A| < \epsilon$, then $L(f;\mathcal{P}') > S(f;\dot{\mathcal{P}}') - \epsilon > A - 2\epsilon$

$$\therefore L(f) = \sup_{P \in \mathcal{P}([a,b])} L(f;P) > A - 2\epsilon$$

$\because \epsilon > 0$ is arbitrary, then $L(f) \ge A$.

Combining ① and ②, $A \le L(f) \le U(f) \le A$, then $L(f) = U(f) = A$. f Darboux integrable.

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