Topic #7 The Fundamental Theorem

Connection between derivative and integral

Plan:

* Fundamental Theorem of Calculus (1^{st} Form & 2^{nd} Form)
* Substitution Theorem
* Lebesgue's Integrability Criterion
* Composition Theorem
* Product Theorem
* Integration by Parts
* Taylor's Theorem with remainder of integral form

Recall : If $F'(x) = f(x), \ \forall x \in [a, b]$
then F is called an antiderivative or a primitive of f on $[a, b]$.

note : one sided derivative at $x=a$ & $x=b$.

Fundamental Theorem of Calculus (1^{st} Form):

Suppose :

(a) F is continuous on $[a, b]$

(b) $F'(x) = f(x), \ \forall x \in [a,b] \setminus E$ for some finite set E in $[a,b]$

(c) $f \in \mathcal{R}[a,b]$

then
$$\int_a^b f = F(b) - F(a).$$

i.e., Consider an integrable function (must be bounded). If up to a finite set it is a derivative of a continuous function, then …

Pf. Assume $E = \{a, b\}$. (general case : Exercise)
then (b) is reduced to
$$F'(x) = f(x), \ \forall x \in (a, b)$$
Let $\epsilon > 0$.

$\because f \in \mathcal{R}[a,b]$

$\therefore$ (*) $\exists \delta_\epsilon > 0$ s.t. $\forall \dot{\mathcal{P}} = \{[x_{i-1}, x_i] ; t_i\}_{i=1}^n$ with $\|\dot{\mathcal{P}}\| < \delta_\epsilon$,
$$|S(f; \dot{\mathcal{P}}) - \int_a^b f| < \epsilon.$$

(arbitrary tags)

(no taggs for the moment)

Now, we take a partition $\mathcal{P}=\{[x_{i-1},x_i]\}_{i=1}^{n}$ with $\|\mathcal{P}\|<\delta_\epsilon$.

Apply Mean Value Theorem to F on $[x_{i-1},x_i]$, then

$\exists\, u_i\in(x_{i-1},x_i)$ s.t.

$$F(x_i)-F(x_{i-1})=F'(u_i)(x_i-x_{i-1})$$
$$=f(u_i)(x_i-x_{i-1}),\quad i=1,2,\cdots,n$$

(note: $F'(x)=f(x),\ \forall x\in(a,b)$).

therefore

$$F(b)-F(a)=\sum_{i=1}^{n}\left[F(x_i)-F(x_{i-1})\right]$$
$$=\sum_{i=1}^{n}f(u_i)(x_i-x_{i-1})$$

Define $\dot{\mathcal{P}}_u=\{[x_{i-1},x_i],\ u_i\}_{i=1}^{n}$ specific taggs

Then, $\|\dot{\mathcal{P}}_u\|<\delta_\epsilon$ and $F(b)-F(a)=S(f;\dot{\mathcal{P}}_u)$

Applying (*),

$$\left|F(b)-F(a)-\int_a^b f\right|<\epsilon.$$

Since $\epsilon>0$ is arbitrary, $\int_a^b f=F(b)-F(a)$. #

Remarks:

(i) If $E=\phi$, then assumption (b) $\Rightarrow$ assumption (a)

(ii) One may allow f defined on $[a,b]$ except at finite number of points in $[a,b]$. In fact, one can extend f to all $x\in[a,b]$ by setting $f(c)=0$ for any $c\notin \text{domain}(f)$ at the beginning.

(iii) Note: F differentiable on $[a,b]$ $\not\Rightarrow$ $F'\in\mathcal{R}[a,b]$.

$\therefore$ assumption (c) is not automatically satisfied.

(even if $E=\phi$ and assumption (b) is satisfied)

Examples:

(a) • $F(x)=\frac{1}{2}x^2$, $x\in[a,b]$; continuous on $[a,b]$

• $F'(x)=x$, $\forall x\in[a,b]$; $\therefore E=\phi$

• $f(x)=x\in\mathcal{R}[a,b]$

By Thm,

$$\int_a^b x\,dx = F(b)-F(a)=\frac{1}{2}b^2-\frac{1}{2}a^2. \quad \#$$

(b) • $G(x)=\text{Arctan}\,x$, $x\in[a,b]$; continuous on $[a,b]$

• $G'(x)=\frac{1}{x^2+1}$, $\forall x\in[a,b]$; $\therefore E=\phi$

• $f(x)=\frac{1}{x^2+1}\in\mathcal{R}[a,b]$ ($\because$ it's continuous)

By Thm,

$$\int_a^b \frac{1}{x^2+1}\,dx = G(b)-G(a)=\text{Arctan}\,b-\text{Arctan}\,a. \quad \#$$

(c) • $A(x)=|x|$, $x\in[-10,10]$; continuous on $[-10,10]$

• Note:

$$A'(x)=\begin{cases}1, & x\in(0,10]\\ \nexists, & x=0\\ -1, & x\in[-10,0).\end{cases}$$

Then, $A'(x)=\text{sgn}(x)$, $\forall x\in[-10,10]\setminus\{0\}$
with $E=\{0\}$.

• Note: $\text{Sgn}(x)$ is a step function (with one degenerated interval)
$\therefore \text{sgn}(x)\in\mathcal{R}[-10,10]$.

By Thm,

$$\int_{-10}^{10}\text{sgn}(x)\,dx = A(10)-A(-10)=10-10=0. \quad \#$$

(d) • $H(x)=2\sqrt{x}$, $x\in[0,b]$; continuous on $[0,b]$

• $H'(x)=\frac{1}{\sqrt{x}}$, $\forall x\in[0,b]\setminus\{0\}$ with $E=\{0\}$.

- Note: $h(x) = \frac{1}{\sqrt{x}}$ is unbounded on $[0,b]$

$\therefore h \notin \mathcal{R}[0,b]$ (no matter how we redefine $H'(0)$)

$\therefore$ <u>Fundamental Thm (1st Form)</u> does NOT apply. #

<u>RK</u>: In this case, need to consider improper integrals, namely, to first apply Fundamental Thm to $[\epsilon, b]$ with $\epsilon > 0$ and then let $\epsilon \to 0$.

(e) Let $K(x) = \begin{cases} x^2 \cos\left(\frac{1}{x^2}\right), & x \in (0,1] \\ 0, & x = 0. \end{cases}$

Compute (Exercise):

$$K'(x) = \begin{cases} 2x\cos\left(\frac{1}{x^2}\right) + \frac{2}{x}\sin\left(\frac{1}{x^2}\right), & x \in (0,1] \\ 0, & x = 0. \end{cases}$$

This gives:

K is differentiable on $[0,1]$,

hence continuous on $[0,1]$;

But, K' is unbounded on $[0,1]$;

$\therefore$ Similar to (d),

$K' \notin \mathcal{R}[0,1]$ and F.T. does NOT apply. #

Def. Let $f \in \mathcal{R}[a,b]$, then $F(z) \overset{\text{def.}}{=} \int_a^z f$ for $z \in [a,b]$ is called the indefinite integral of f with basepoint a.

Thm If $f \in \mathcal{R}[a,b]$, then $F(z) = \int_a^z f$ is Lipschitz continuous on $[a,b]$, namely,

$$\exists\, L \geq 0 \text{ s.t. } |F(z) - F(w)| \leq L|z - w|, \quad \forall z, w \in [a,b].$$

Pf. Let $z, w \in [a,b]$ with $w \leq z$ without loss of generality.

$$a \quad w \quad z \quad b$$

Additivity Thm gives:

$$F(z) = \int_a^z f = \int_a^w f + \int_w^z f = F(w) + \int_w^z f$$

$$\therefore \; F(z) - F(w) = \int_w^z f.$$

$\because f \in \mathcal{R}[a,b]$

$\therefore f$ is bounded on $[a,b]$,

i.e. $\exists M \geq 0$ s.t. $|f(x)| \leq M, \; \forall x \in [a,b]$;

i.e. $-M \leq f(x) \leq M, \; \forall x \in [a,b]$.

Hence

$$-M(z-w) \leq \int_w^z f \leq M(z-w)$$

i.e. $\left| \int_w^z f \right| \leq M|z-w| \quad (z-w \geq 0)$

Therefore

$$|F(z) - F(w)| = \left| \int_w^z f \right| \leq M|z-w|. \qquad \#$$

next goal: indefinite integral F is differentiable at any point where f is continuous.

Fundamental Theorem of Calculus (2nd Form):
Let $f \in \mathcal{R}[a,b]$ which is continuous at $c \in [a,b]$,
then $F(z) = \int_a^z f$ is differentiable at $z = c$ and
$F'(c) = f(c)$.

Pf: Only consider $c \in [a,b)$ and the right-hand derivative

$$\lim_{h \to 0^+} \frac{F(c+h) - F(c)}{h} = f(c). \quad \text{———} \quad (*)$$

proof in case of the left-hand derivative at $c \in (a, b]$ is similar.

Now, we take $c \in [a, b)$ and consider the proof of $(*)$.

Let $\epsilon > 0$.

$\because$ f is continuous at c

$\boxed{(**)}$ $\therefore$ $\exists \eta_\epsilon > 0$ s.t. $|f(x) - f(c)| < \epsilon$, $\forall x \in [c, c+\eta_\epsilon)$.

Let $h \in (0, \eta_\epsilon)$. Consider $(\subset [a,b])$

$$a \leq c < c+h \leq b.$$

Additivity Thm gives

$f \in \mathcal{R}[a, c+h]$, $\mathcal{R}[a, c]$, and $\mathcal{R}[c, c+h]$

with
$$\int_a^{c+h} f = \int_a^c f + \int_c^{c+h} f$$

that's
$$F(c+h) = F(c) + \int_c^{c+h} f$$

Apply $(**)$,
$$[f(c) - \epsilon]\, h \leq F(c+h) - F(c) = \int_c^{c+h} f \leq [f(c)+\epsilon]\, h$$

i.e.
$$f(c) - \epsilon \leq \frac{F(c+h) - F(c)}{h} \leq f(c) + \epsilon$$

i.e.
$$\left| \frac{F(c+h) - F(c)}{h} - f(c) \right| \leq \epsilon, \quad \forall h \in (0, \eta_\epsilon).$$

this shows $(*)$. #.

Thm. Let f be continuous on $[a, b]$, then $F(x) \overset{\text{def.}}{=} \int_a^x f$ is differentiable on $[a, b]$ with
$$F'(x) = f(x), \quad \forall x \in [a, b].$$
(i.e. indefinite integral F is an antiderivative of f on $[a,b]$)

Pf: f continuous on $[a, b] \Rightarrow f \in \mathcal{R}[a, b]$

and f continuous at any p't of $[a, b]$

$\xrightarrow[\text{(2nd Form)}]{\text{F.T.C.}}$ conclusion. #

Examples :

(a) $f(x) \overset{\text{def.}}{=} \operatorname{sgn}(x)$, $x \in [-1, 1]$.

then $f \in \mathcal{R}[-1, 1]$ ($\because$ it's a step function)

and (Exercise) :

$$F(x) \overset{\text{def.}}{=} \int_{-1}^{x} \operatorname{sgn}(x)\, dx = |x| - 1 .$$

Note : $\because$ $F'(0)$ does not exist

$\therefore$ F can NOT be an antiderivative of f on $[-1,1]$ #

(b) Again recall Thomae's function :

$$h(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \in [0,1] \text{ with } m, n \in \{1, 2, \cdots\} \text{ and g.c.d.}(m,n) = 1. \\ 1, & \text{if } x = 0 \\ 0, & \text{if } x \text{ is irrational in } [0, 1]. \end{cases}$$

We proved (before) that

$$0 \le h \in \mathcal{R}[0, 1] \text{ with } \int_0^1 h = 0.$$

$\therefore$ $H(x) \overset{\text{def.}}{=} \int_0^x h = 0$, $\forall x \in [0, 1]$.

direct to see : $H'(x) = 0$, $\forall x \in [0, 1]$.

But, $H'(x) \neq h(x)$, $\forall x$ rational in $[0, 1]$.

In this case,

H can NOT be an antiderivative of h on $[0, 1]$. #

Two examples above are consistent with the Thm :

Whenever indefinite integral F is not an antiderivative of f on $[a, b]$, f may NOT be continuous at some points of $[a, b]$.

(a) : $f(x) = \operatorname{sgn}(x)$ not continuous at $x = 0$

(b) : $h(x)$ is NOT continuous at any rational point of $[0, 1]$. #

Thm (Substitution Theorem)
Let $\varphi: [\alpha, \beta] \to \mathbb{R}$ have a continuous derivative on $[\alpha, \beta]$ and $f: I \to \mathbb{R}$ be continuous on I with $\varphi([\alpha, \beta]) \subset I$, then

$$\int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t)\, dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\, dx$$

Pf. Exercise.

Rks: (i) It says a method of "chang of variable" to evaluate integrals:
$x = \varphi(t)$, $dx = \varphi'(t)\, dt$.

(ii) On RHS, it allows for $\varphi(\alpha) \geq \varphi(\beta)$.

Lebesgue's Integrability Criterion

a necessary & sufficient condition for a bounded function on an interval to be Riemann integrable!

Def:
(a) A set $Z \subset \mathbb{R}$ is a null set if $\forall \epsilon > 0$, $\exists$ a countable collection $\{(a_k, b_k)\}_{k=0}^{\infty}$ of open intervals (allowed to be overlapped) s.t.

$$Z \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k) \quad \text{and} \quad \sum_{k=1}^{\infty} (b_k - a_k) \leq \epsilon.$$

(b) Let $Q(x)$ be a statement about the point $x \in I$.
We say "$Q(x)$ holds almost everywhere on I (or for almost every $x \in I$)"
if $\exists$ a null set $Z \subset I$ s.t.
$Q(x)$ holds for all $x \in I \setminus Z$.
In this case, we write:
"$Q(x)$ for a.e. $x \in I$."

Remarks:

(i) A null set is a set of measure zero in the theory of integration. A null set means that it can be covered by the union of countable open intervals with total length as small as we wish.

(ii) Any subset of a null set is a null set; The union of two null sets is a null set.

Example:

$\mathbb{Q}_1 \overset{\text{def.}}{=} \{$ rational numbers in $[0,1]\}$

then $\mathbb{Q}_1$ is a null set

Pf. $\because$ $\mathbb{Q}_1$ is countable

$\therefore$ We can write $\mathbb{Q}_1 = \{r_1, r_2, \cdots\}$

Let $\epsilon > 0$. Define open intervals:

$$J_k = \left(r_k - \frac{\epsilon}{2^{k+1}},\ r_k + \frac{\epsilon}{2^{k+1}}\right),\ k = 1, 2, \cdots$$

Obvious to see: $r_k \in J_k$, length of $J_k = \dfrac{\epsilon}{2^k}$.

$\therefore$ $\mathbb{Q}_1 \subset \bigcup_{k=1}^{\infty} J_k$ and $\sum_{k=1}^{\infty}(\text{length of } J_k) = \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon$.

Hence, $\mathbb{Q}_1$ is a null set. #

Note ① Proof above also implies:

Every countable set is a null set

② There exists a null set which is uncoutable.

example: the Cantor set (Chap 11)

Thm (Lebesgue's Integrability Criterion)

Let $f: [a,b] \to \mathbb{R}$ be bouned. Then, $f \in \mathcal{R}[a,b]$ iff f is continuous almost everywhere on $[a,b]$.

Pf: Omitted. See App. C of the Textbook.

Examples:

(a) Every step function on $[a,b]$ is Riemann integrable.

In fact, every step function is bounded

& has at most a finite set of points of discontinuity.

$\therefore$ Lebesgue's Integrability Criterion is verified. #

(b) Every monotone function on $[a,b]$ is Riemann integrable.

In fact, * monotone function is bounded.

* Chap 5 (Thm 6.4) tells:

set of points of discontinuity of a monotone function is countable.

$\therefore$ L.I.C. is verified. #

(c) Revisit example: $G(x) \overset{\text{def.}}{=} \begin{cases} \frac{1}{n}, & \text{if } x = \frac{1}{n} \ (n=1,2,\cdots) \\ 0, & \text{elsewhere on } [0,1] \end{cases}$

$\in \mathcal{R}[0,1]$.

In fact, * G is bounded on $[0,1]$.

* $\{x \in [0,1] : G \text{ discontinuous at } x\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$

countable, so ~~is~~ is a null set.

$\therefore$ L.I.C. verified. #

(d) Revisit Dirichlet function: $f(x) = \begin{cases} 1 & \text{if } x \text{ rational in } [0,1] \\ 0 & \text{if } x \text{ irrational in } [0,1] \end{cases}$

$\notin \mathcal{R}[0,1]$

In fact, * f is bounded on $[0,1]$

* f is discontinuous at any $x \in [0,1]$

$\therefore$ set of discontinuity $= [0,1]$

which is NOT a null set.

(Exercise!)

$\therefore$ L.I.C. gives: $f \notin \mathcal{R}[0,1]$. #

(e) Revisit Thomae's function

$$h(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \in [0,1] \text{ with } m, n \in \{1, 2, \cdots\} \ \& \ \text{g.c.d.}(m,n) = 1 \\ 1, & \text{if } x = 0 \\ 0 & \text{if } x \text{ irrational in } [0,1]. \end{cases}$$

$\in \mathcal{R}[0,1]$.

In fact, * h is bounded on $[0,1]$

* set of discontinuity $= \{$rational numbers in $[0,1]\}$ $= \mathbb{Q}_1$, is a null set.

$\therefore$ L.I.C. is verified. #

Thm (Composition Theorem)

Let • $f \in \mathcal{R}[a,b]$ with $f([a,b]) \subseteq [c,d]$

• $\varphi : [c,d] \to \mathbb{R}$ continuous.

then $\varphi \circ f \in \mathcal{R}[a,b]$.

Pf: Note: If f is continuous at a point $x \in [a,b]$ then $\varphi \circ f$ is also continuous at x. ($\because \varphi$ is continuous on $[c,d]$). (*)

Let

$D \stackrel{\text{def.}}{=}$ set of points of discontinuity of f

$\because f \in \mathcal{R}[a,b]$

$\therefore f$ is bounded, and Lebesgue's Integrability Criterion tells: D is a null set.

Let

$D_1 \stackrel{\text{def}}{=}$ set of points of discontinuity of $\varphi \circ f$

$\because$ (*) tells: $\{x \in [a,b] : x \notin D\} \subseteq \{x \in [a,b] ; x \notin D_1\}$

$\therefore D_1 \subseteq D$, then D_1 is also a null set.

Note: $\varphi \circ f$ is bounded on $[a,b]$ ($\because f$ bounded on $[a,b]$ and φ is continuous on $[c,d]$)

~~therefore~~, Lebesgue's Integrability Criterion gives:

$\varphi \circ f \in \mathcal{R}[a,b]$. #

Rk: "φ continuous" is essential; see Ex. 22. #

Cor Let $f \in \mathcal{R}[a,b]$, then $|f| \in \mathcal{R}[a,b]$ and
$$\left|\int_a^b f\right| \leq \int_a^b |f| \leq M(b-a)$$
where $M \geq 0$ is a constant s.t. $|f(x)| \leq M$, $\forall x \in [a,b]$.

Pf. $\because f \in \mathcal{R}[a,b]$

$\therefore f$ is bounded on $[a,b]$

Let $M \geq 0$ be s.t. $|f(x)| \leq M$, $\forall x \in [a,b]$.

Note: • $f([a,b]) \subseteq [-M, M]$

• $|\cdot| : [-M, M] \to \mathbb{R}$ is continuous

Then, Composition Thm gives: $|f| \in \mathcal{R}[a,b]$.

$\because -|f|(x) \leq f(x) \leq |f|(x)$, $\forall x \in [a,b]$

$$\therefore \int_a^b [-|f|(x)]\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f|(x)\,dx$$

$$\text{i.e. } -\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|$$

$$\text{i.e. } \left|\int_a^b f\right| \leq \int_a^b |f|$$

Similarly, "$|f|(x) \leq M$, $\forall x \in [a,b]$" gives
$$\int_a^b |f| \leq \int_a^b M = M(b-a).$$
#

Thm (Product Theorem)
Let $f, g \in \mathcal{R}[a,b]$, then $fg \in \mathcal{R}[a,b]$.

Pf: $\because f \in \mathcal{R}[a,b]$

$\therefore \exists M \geq 0$ s.t. $f([a,b]) \subseteq [-M, M]$

note: $\varphi(t) \overset{\text{def.}}{=} t^2 : [-M, M] \to \mathbb{R}$ continuous

$\therefore f^2 \in \mathcal{R}[a,b]$

Similarly, "$g \in \mathcal{R}[a,b]$" gives: $g^2 \in \mathcal{R}[a,b]$.

Moreover, $f, g \in \mathcal{R}[a,b] \Rightarrow f+g \in \mathcal{R}[a,b]$

$\Rightarrow (f+g)^2 \in \mathcal{R}[a,b]$.

Then, $fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right] \in \mathcal{R}[a,b]$. #

Thm (Integration by Parts)

Let
- F, G be differentiable on $[a,b]$
- $f = F'$, $g = G' \in \mathcal{R}[a,b]$

then

$fG, Fg \in \mathcal{R}[a,b]$ and

$$\int_a^b fG = FG\Big|_a^b - \int_a^b Fg$$

namely, $\int_a^b (F'G) = FG\Big|_a^b - \int_a^b (FG')$.

Pf. $\because$ F, G are differentiable on $[a,b]$

$\therefore$ F, G are continuous on $[a,b]$

and FG is differentiable on $[a,b]$ with

$(*)$ ——— $(FG)' = F'G + FG'$.

Product thm

Note: $F' \in \mathcal{R}[a,b]$, $G \in \mathcal{R}[a,b]$ ($\because$ continuous) $\Rightarrow$ $F'G \in \mathcal{R}[a,b]$

Similarly

$G' \in \mathcal{R}[a,b]$, $F \in \mathcal{R}[a,b]$ ($\because$ continuous) $\Rightarrow$ $FG' \in \mathcal{R}[a,b]$

therefore, $(*)$ gives: $(FG)' = F'G + FG' \in \mathcal{R}[a,b]$.

By Fundamental Thm of Calculus (1^{st} Form),

$$\int_a^b (F'G + FG') = FG\Big|_a^b$$

i.e. $\int_a^b (F'G) = FG\Big|_a^b - \int_a^b (FG')$. #

Thm (Taylor's Theorem with Remainder of Integral Form)

Let
- $f: [a,b] \to \mathbb{R}$
- $f', \cdots, f^{(n)}, f^{(n+1)}$ exist on $[a,b]$
- $f^{(n+1)} \in \mathcal{R}[a,b]$.

then

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + R_n$$

with

$$R_n = \frac{1}{n!}\int_a^b f^{(n+1)}(t)\,(b-t)^n\,dt.$$

Pf: $R_n = \frac{1}{n!}\int_a^b f^{(n+1)}(t)\,(b-t)^n\,dt$ (well-defined by Product Thm)

$$= \int_a^b (f^{(n)})'(t)\left(\frac{(b-t)^n}{n!}\right)dt$$

$$= f^{(n)}(t)\frac{(b-t)^n}{n!}\Big|_a^b - \int_a^b f^{(n)}(t)\left[-\frac{(b-t)^{n-1}}{(n-1)!}\right]dt$$

(Integration by Parts)

$$= -\frac{f^{(n)}(a)}{n!}(b-a)^n + \underbrace{\frac{1}{(n-1)!}\int_a^b f^{(n)}(t)\,(b-t)^{n-1}\,dt}_{=R_{n-1}}$$

$$= -\frac{f^{(n)}(a)}{n!}(b-a)^n - \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + R_{n-2}$$

(repeated calculation)

$\vdots$ (iteration)

$$= -\left[\frac{f^{(n)}(a)}{n!}(b-a)^n + \cdots + \frac{f'(a)}{1!}(b-a)\right] + R_0,$$

where

$$R_0 = \frac{1}{0!}\int_a^b f'(t)(b-t)^0\,dt = \int_a^b f'(t)\,dt = f(b) - f(a).$$

#

— END —