Topic #6 Riemann Integrable Functions

Plan:

* Cauchy Criterion
* Squeeze Theorem and $\left.\begin{array}{l}\text{step functions}\\ \text{continuous functions}\\ \text{monotone functions}\end{array}\right\}$ are Riemann integrable
* Additive property

Thm (Cauchy Criterion for Riemann Integrability)

$f \in \mathcal{R}[a,b]$ iff $\forall \epsilon > 0$, $\exists \eta_\epsilon > 0$ s.t.

if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are any two tagged partitions of $[a,b]$ with $\|\dot{\mathcal{P}}\| < \eta_\epsilon$ and $\|\dot{\mathcal{Q}}\| < \eta_\epsilon$,

then

$$|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| < \epsilon. \quad \#$$

Recall: (x_n) converges $\Leftrightarrow \forall \epsilon > 0$, $\exists N_\epsilon > 0$ s.t.

if $m, n \geq N_\epsilon$, then $|x_m - x_n| < \epsilon$.

Pf of Thm:

$(\overset{\text{only if}}{\Longrightarrow})$ Assume $f \in \mathcal{R}[a,b]$. Let $L := \int_a^b f \in \mathbb{R}$.

Let $\epsilon > 0$, then $\exists \eta_\epsilon > 0$ s.t.

$$|S(f;\dot{\mathcal{P}}) - L| < \epsilon/2 \quad \forall \dot{\mathcal{P}} \text{ with } \|\dot{\mathcal{P}}\| < \eta_\epsilon.$$

$$\cancel{|S(f;\dot{\mathcal{Q}}) - L| < \epsilon/2}$$

Let $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ be two tagged partitions of $[a,b]$

with $\|\dot{\mathcal{P}}\| < \eta_\epsilon$ and $\|\dot{\mathcal{Q}}\| < \eta_\epsilon$,

then

$$|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| \leq |S(f;\dot{\mathcal{P}}) - L| + |S(f;\dot{\mathcal{Q}}) - L|$$
$$< \epsilon/2 + \epsilon/2 = \epsilon. \quad \#$$

("$\stackrel{\text{if}}{\Leftarrow}$") By the assumption, for $n \in \mathbb{N} = \{1, 2, 3, \cdots\}$, $\exists \eta_n > 0$ s.t.

$$|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| < \frac{1}{n} \quad \forall \dot{\mathcal{P}}, \dot{\mathcal{Q}} \text{ with } \|\dot{\mathcal{P}}\| < \eta_n, \ \|\dot{\mathcal{Q}}\| < \eta_n.$$

Define

$$\delta_n = \min\{\eta_1, \cdots, \eta_n\} > 0 \quad (n = 1, 2, \cdots)$$

Note: $\delta_n \geq \delta_{n+1}$ for $n \in \mathbb{N}$, and

$$|S(f;\dot{\mathcal{P}}) - S(f,\dot{\mathcal{Q}})| < \frac{1}{n}, \forall \dot{\mathcal{P}}, \dot{\mathcal{Q}} \text{ with } \|\dot{\mathcal{P}}\| < \delta_n, \ \|\dot{\mathcal{Q}}\| < \delta_n. \qquad \text{——— } (*)$$

For each $n \in \mathbb{N}$,

choose $\dot{\mathcal{P}}_n$ to be a tagged partition of $[a, b]$ with $\|\dot{\mathcal{P}}_n\| < \delta_n$ (it exist!)

<u>Claim</u>: $(S(f;\dot{\mathcal{P}}_n))_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$,

so $\exists L \in \mathbb{R}$ s.t. $L = \lim_{n\to\infty} S(f;\dot{\mathcal{P}}_n)$

<u>Pf of claim:</u>

Let $m > n \geq 1$.

Note: $\|\dot{\mathcal{P}}_n\| < \delta_n$, $\|\dot{\mathcal{P}}_m\| < \delta_m \leq \delta_n$

Then, by $(*)$,

$$|S(f;\dot{\mathcal{P}}_n) - S(f;\dot{\mathcal{P}}_m)| < \frac{1}{n}. \qquad \text{——— } (**)$$

So, $(S(f;\dot{\mathcal{P}}_n))_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. #

Take $m \to \infty$ in $(**)$, gives

$$|S(f;\dot{\mathcal{P}}_n) - L| < \frac{1}{n}, \quad \forall n \in \mathbb{N}. \qquad \text{——— } (***)$$

To show: $f \in \mathcal{R}[a,b]$ with $\int_a^b f = L$.

In fact, let $\epsilon > 0$, then take N s.t. $N > \frac{2}{\epsilon}$.

For any $\dot{\mathcal{Q}}$ with $\|\dot{\mathcal{Q}}\| < \delta_N$. ($\delta_N > 0$),

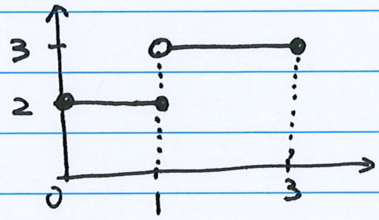
$$|S(f;\dot{\mathcal{Q}}) - L| \leq |S(f;\dot{\mathcal{Q}}) - S(f;\dot{\mathcal{P}}_N)| + |S(f;\dot{\mathcal{P}}_N) - L|$$
$$< \underset{\text{(use }(*))}{\frac{1}{N}} + \underset{\text{(use }(***))}{\frac{1}{N}} = \frac{2}{N} < \epsilon$$

By def. $f \in \mathcal{R}[a,b]$ and $L = \int_a^b f$. ##

Examples as quick applications:

(a) Let $g:[0,3]\to\mathbb{R}$ be defined by

$$g(x)=\begin{cases}3, & 1<x\le 3\\ 2, & 0\le x\le 1.\end{cases}$$



Use Cauchy Criterion to show:

g is Riemann integrable on $[0,3]$.

Pf: As in topic #5, we showed:

Let $\delta>0$, then if $\|\dot{\mathcal{P}}\|<\delta$, then

$$8-5\delta\le S(g;\dot{\mathcal{P}})\le 8+5\delta$$

Let $\dot{Q}$ be another tagged partition of $[0,3]$ with $\|\dot{Q}\|<\delta$, then still have:

$$8-5\delta\le S(g;\dot{Q})\le 8+5\delta.$$

Hence, $|S(g;\dot{\mathcal{P}})-S(g;\dot{Q})|\le(8+5\delta)-(8-5\delta)=10\delta$.

Now, let $\epsilon>0$, define $\eta_\epsilon=\frac{\epsilon}{20}>0$,

If $\dot{\mathcal{P}},\dot{Q}$ are two tagged partitions with $\|\dot{\mathcal{P}}\|<\eta_\epsilon$, $\|\dot{Q}\|<\eta_\epsilon$,

then

$$|S(g;\dot{\mathcal{P}})-S(g;\dot{Q})|\le 10\cdot\eta_\epsilon=10\cdot\frac{\epsilon}{20}=\frac{\epsilon}{2}<\epsilon.$$

Therefore, Cauchy Criterion is verified. #

(b) Cauchy Criterion is also a criterion that a function is NOT integrable, i.e.

$f\notin\mathcal{R}[a,b]\iff\exists\,\epsilon_0>0$ s.t. $\forall\,\eta>0$,

$\exists\,\dot{\mathcal{P}},\dot{Q}$ with $\|\dot{\mathcal{P}}\|<\eta$ & $\|\dot{Q}\|<\eta$ s.t.

$|S(f;\dot{\mathcal{P}})-S(f;\dot{Q})|\ge\epsilon_0$. #

An example:

Dirichlet function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ rational}, \in [0,1] \\ 0, & \text{if } x \text{ irrational}, \in [0,1] \end{cases}.$$

Show: $f \notin \mathcal{R}[0,1]$.

pf: Let $\epsilon_0 = 1/2$. Let $\eta > 0$ be arbitrary.
Let $\dot{\mathcal{P}}$ be a tagged partition with rational tags s.t. $\|\dot{\mathcal{P}}\| < \eta$
$\dot{Q}$ be a tagged partition with irrational tags s.t. $\|\dot{Q}\| < \eta$.

Compute:

$$S(f;\dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

rational

$$= \sum_{i=1}^{n} (x_i - x_{i-1})$$

$$= 1,$$

and

$$S(f;\dot{Q}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

irrational

$$= 0.$$

Therefore, for such $\dot{\mathcal{P}}$ and $\dot{Q}$,

$$|S(f;\dot{\mathcal{P}}) - S(f;\dot{Q})| = 1 \geq \epsilon_0.$$

By Cauchy Criterion,

$$f \notin \mathcal{R}[0,1]. \quad \#$$

More applications of Cauchy Criterion:

Thm (Squeeze Theorem) $f \in \mathcal{R}[a,b]$ iff $\forall \epsilon > 0$, $\exists \alpha_\epsilon, \omega_\epsilon \in \mathcal{R}[a,b]$ with $\alpha_\epsilon(x) \leq f(x) \leq \omega_\epsilon(x)$, $\forall x \in [a,b]$ s.t. $\int_a^b (\omega_\epsilon - \alpha_\epsilon) < \epsilon$.

Pf: ($\Rightarrow$) Let $\epsilon > 0$. Just take $\alpha_\epsilon = \omega_\epsilon = f$.

($\Leftarrow$) Let $\epsilon > 0$.

Let functions $\alpha_\epsilon, \omega_\epsilon \in \mathcal{R}[a,b]$ with

$$\alpha_\epsilon(x) \le f(x) \le \omega_\epsilon(x), \quad \forall x \in [a,b] \quad (*)$$

such that

$$\int_a^b (\omega_\epsilon - \alpha_\epsilon) < \epsilon.$$

$\because$ $\alpha_\epsilon, \omega_\epsilon \in \mathcal{R}[a,b]$.

$\therefore$ $\exists \delta_\epsilon > 0$ s.t.

$$\left.\begin{aligned} \left| S(\alpha_\epsilon; \dot{\mathcal{P}}) - \int_a^b \alpha_\epsilon \right| &< \epsilon \\ \left| S(\omega_\epsilon; \dot{\mathcal{P}}) - \int_a^b \omega_\epsilon \right| &< \epsilon \end{aligned}\right\} \forall \dot{\mathcal{P}} \text{ with } \|\dot{\mathcal{P}}\| < \delta_\epsilon. \quad (**)$$

Let $\dot{\mathcal{P}}, \dot{\mathcal{Q}}$ be tagged partitions with $\|\dot{\mathcal{P}}\| < \delta_\epsilon$, $\|\dot{\mathcal{Q}}\| < \delta_\epsilon$.

(**) tells:

$$\int_a^b \alpha_\epsilon - \epsilon < S(\alpha_\epsilon; \dot{\mathcal{P}})$$

$$S(\omega_\epsilon; \dot{\mathcal{P}}) < \int_a^b \omega_\epsilon + \epsilon$$

Note also: (*) tells:

$$S(\alpha_\epsilon; \dot{\mathcal{P}}) \le S(f; \dot{\mathcal{P}}) \le S(\omega_\epsilon; \dot{\mathcal{P}})$$

Combing both,

$$\int_a^b \alpha_\epsilon - \epsilon < S(f; \dot{\mathcal{P}}) < \int_a^b \omega_\epsilon + \epsilon$$

In the same way,

$$\int_a^b \alpha_\epsilon - \epsilon < S(f; \dot{\mathcal{Q}}) < \int_a^b \omega_\epsilon + \epsilon$$

Therefore,

$$|S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}})| < \left(\int_a^b \omega_\epsilon + \epsilon\right) - \left(\int_a^b \alpha_\epsilon - \epsilon\right)$$

$$= \int_a^b (\omega_\epsilon - \alpha_\epsilon) + 2\epsilon < 3\epsilon.$$

Since $\epsilon > 0$ is arbitrary, Cauchy Criterion implies $f \in \mathcal{R}[a,b]$. #

to show: "step function" $\in \mathcal{R}[a,b]$.

Recall:

$\varphi : [a,b] \to \mathbb{R}$ is a step function
if $\exists \{k_i\}_{i=1}^{n}$ (set of finite, distinct numbers) s.t.

$$\varphi|_{I_i} \equiv k_i$$

where I_i are subintervals of $[a,b]$ s.t.
(not necessary closed)

$$[a,b] = \bigcup_{i=1}^{n} I_i \quad \text{with } I_i \cap I_j = \phi \text{ for } i \neq j$$

(disjoint union)

(NOTE: A number k_i may be attained by φ on one or more subintervals.).

Lemma Let J be a subinterval of $[a,b]$ with end points $c<d$.
Define

$$\varphi_J(x) = \begin{cases} 1 & \text{if } x \in J \\ 0 & \text{if } x \notin J \text{ with } x \in [a,b] \end{cases}$$

Then,

$$\varphi_J \in \mathcal{R}[a,b] \text{ with } \int_a^b \varphi_J = d-c.$$

Pf: Four cases for J:

$J = [c,d], (c,d], [c,d), (c,d)$.

and correspondingly four cases for $\varphi_J(x)$, $a \le x \le b$.

Case 1: $J = [c,d]$.

Ex 7.1.13 (tutorial) tells: $\varphi_J \in \mathcal{R}[a,b]$ with $\int_a^b \varphi_J = d-c$.

In the latter three cases, functions have the same values as $\varphi_{[c,d]}$ except at finitely many points, hence these functions $\varphi_{(c,d]}$, $\varphi_{[c,d)}$ and $\varphi_{(c,d)}$ are also in $\mathcal{R}[a,b]$ with the same integral $= d-c$. #

Thm Let $\varphi: [a,b] \to \mathbb{R}$ be a step function, then $\varphi \in \mathcal{R}[a,b]$.

pf: Recall the def of a step function:

$$\varphi|_{I_i} = k_i \quad (1 \le i \le n) \quad \text{with } [a,b] = \underset{(\text{disjoint})}{\bigcup} I_i$$

For each $i \in \{1, \cdots, n\}$, introduce $\varphi_{I_i}(x)$, $a \le x \le b$ as in the previous lemma.

Therefore

$$\varphi(x) = \sum_{i=1}^{n} k_i \varphi_{I_i}(x), \quad \forall x \in [a,b].$$

By Lemma, $\varphi_{I_i} \in \mathcal{R}[a,b]$, $1 \le i \le n$,
then their linear combination is also in $\mathcal{R}[a,b]$.
so $\varphi \in \mathcal{R}[a,b]$. #

RK: If each I_i has end points $c_i < d_i$, then

$$\int_a^b \varphi = \int_a^b \sum_{i=1}^{n} k_i \varphi_{I_i}$$

$$= \sum_{i=1}^{n} k_i \int_a^b \varphi_{I_i}$$

$$= \sum_{i=1}^{n} k_i (d_i - c_i). \quad \#$$

Examples:

(a) Revisit $g: [0,3] \to \mathbb{R}$ by $g(x) = \begin{cases} 3, & 1 < x \le 3 \\ 2, & 0 \le x < 1 \end{cases}$.

It is a step function:

$$g(x) = 2\varphi_{[0,1)}(x) + 3\varphi_{(1,3]}(x), \quad 0 \le x \le 3.$$

Then $g \in \mathcal{R}[0,3]$ with

$$\int_0^3 g = 2\int_0^3 \varphi_{[0,1)} + 3\int_0^3 \varphi_{(1,3]} = 2\cdot(1-0) + 3\cdot(3-1)$$

$$= 8. \quad \#$$

(b) Re-visit the example:

$$h(x) = x, \quad 0 \le x \le 1$$

show: $h \in \mathcal{R}[0,1]$ with $\int_0^1 h = \frac{1}{2}$.

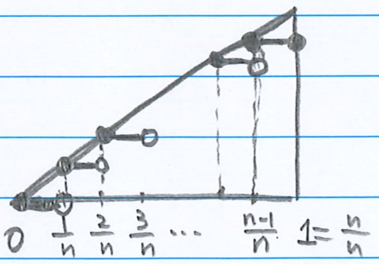
Pf (use the Squeeze Theorem):
Introduce the uniform partition of $[0,1]$:

$$\mathcal{P}_n = \left\{ \left[\frac{k-1}{n}, \frac{k}{n}\right] \right\}_{k=1}^{n}$$

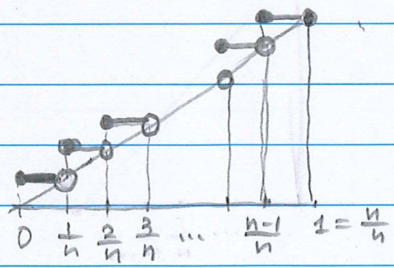
Define two step functions $\begin{cases} \alpha_n : [0,1] \to \mathbb{R} \\ \omega_n : [0,1] \to \mathbb{R} \end{cases}$ by

$$\alpha_n(x) = \begin{cases} \frac{k-1}{n}, & \text{for } x \in \left[\frac{k-1}{n}, \frac{k}{n}\right) \text{ with } k = 1, \cdots, n \\ \frac{n-1}{n} & \text{for } x = 1 \end{cases}$$

$$\omega_n(x) = \begin{cases} \frac{k}{n}, & \text{for } x \in \left[\frac{k-1}{n}, \frac{k}{n}\right) \text{ with } k = 1, \cdots, n \\ 1, & \text{for } x = 1 \end{cases}$$



$\alpha_n(x) \le x = h(x)$
$0 \le x \le 1$



$h(x) = x \le \omega_n(x)$
$0 \le x \le 1$

Obvious to see: $\alpha_n(x) \le h(x) \le \omega_n(x)$, $0 \le x \le 1$.
Furthermore, $\alpha_n \in \mathcal{R}[0,1]$ and $\omega_n \in \mathcal{R}[0,1]$ with

$$\int_0^1 \alpha_n = 0 \cdot \frac{1}{n} + \frac{1}{n} \cdot \frac{1}{n} + \frac{2}{n} \cdot \frac{1}{n} + \cdots + \frac{n-1}{n} \cdot \frac{1}{n}$$

$$= \frac{1}{n^2}\left[0 + 1 + 2 + \cdots + (n-1)\right]$$

$$= \frac{1}{n^2} \frac{n \cdot (n-1)}{2} = \frac{1}{2}\left(1 - \frac{1}{n}\right).$$

and

$$\int_0^1 w_n = \frac{1}{n}\cdot\frac{1}{n} + \frac{2}{n}\cdot\frac{1}{n} + \frac{3}{n}\cdot\frac{1}{n} + \cdots + \frac{n}{n}\cdot\frac{1}{n}$$

$$= \frac{1}{n^2}\left(1+2+3+\cdots+n\right)$$

$$= \frac{1}{n^2}\cdot\frac{(1+n)\,n}{2} \quad = \frac{1}{2}\left(1+\frac{1}{n}\right).$$

thus

$$\int_0^1 (w_n - \alpha_n) = \int_0^1 w_n - \int_0^1 \alpha_n = \frac{1}{n}.$$

Now, let $\epsilon > 0$ be arbitrary.

Choose n_ϵ such that $\frac{1}{n_\epsilon} < \epsilon$,

then we have:

$\alpha_{n_\epsilon}, w_{n_\epsilon} \in R[0,1]$ with

$$\alpha_{n_\epsilon}(x) \le h(x) \le w_{n_\epsilon}(x), \quad \forall x \in [0,1]$$

$$\int_0^1 (w_{n_\epsilon} - \alpha_{n_\epsilon}) = \frac{1}{n_\epsilon} < \epsilon.$$

By Squeeze Theorem,

$$h(x) = x \in \mathcal{R}[0,1].$$

$\boxed{\text{RK}}$: The proof tells: $\forall \epsilon > 0$, $\exists\, n_\epsilon$ with $n_\epsilon > \frac{1}{\epsilon}$ such that

$$\frac{1}{2}\left(1-\frac{1}{n_\epsilon}\right) \le \int_0^1 h \le \int_0^1 w_{n_\epsilon} = \frac{1}{2}\left(1+\frac{1}{n_\epsilon}\right)$$

letting $\epsilon \to 0$ and hence $n_\epsilon \to \infty$, we get

$$\int_0^1 h = \frac{1}{2}. \quad \#$$

Further to show: $-\infty < a < b < \infty$

① Any continuous function on $[a,b]$ is Riemann integrable.

② Any monotone function on $[a,b]$ is Riemann integrable.

Strategy of the proof:

Squeeze Theorem + Construct step functions.

Thm Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ with $-\infty<a<b<\infty$, then $f\in\mathcal{R}[a,b]$.

Pf: Let $\epsilon>0$.

$\because$ f is continuous on the bounded & closed interval $[a,b]$

$\therefore$ f is uniformly continuous on $[a,b]$,

then $\exists\, \delta=\delta_\epsilon>0$ s.t.

$$|f(x)-f(y)|<\frac{\epsilon}{b-a},\quad \forall\, x,y\in[a,b] \text{ with } |x-y|<\delta.$$

Consider a partition $\mathcal{P}=\{[x_{i-1},x_i]\}_{i=1}^n$ of $[a,b]$ with $\|\mathcal{P}\|<\delta$.

$\because$ f is continuous on $[a,b]$

$\therefore$ $\exists\, t_i'\in[x_{i-1},x_i]$ s.t. $f(t_i')=\min\limits_{[x_{i-1},x_i]} f(x)$

and

$\exists\, t_i''\in[x_{i-1},x_i]$ s.t. $f(t_i'')=\max\limits_{[x_{i-1},x_i]} f(x)$

for $i=1,\cdots,n$.

Define two step functions:

$$\alpha_\epsilon(x)=\begin{cases} f(t_i'), & x\in[x_{i-1},x_i) \quad i=1,\cdots,n-1 \\ f(t_n'), & x\in[x_{n-1},x_n] \end{cases}$$

and

$$w_\epsilon(x)=\begin{cases} f(t_i''), & x\in[x_{i-1},x_i), \quad i=1,\cdots,n-1 \\ f(t_n''), & x\in[x_{n-1},x_n]. \end{cases}$$

Obvious to see:

① $\alpha_\epsilon\in\mathcal{R}[a,b]$ and $w_\epsilon\in\mathcal{R}[a,b]$ satisfy

$$\alpha_\epsilon(x)\le f(x)\le w_\epsilon(x),\qquad \forall\, x\in[a,b].$$

② $$\int_a^b (w_\epsilon-\alpha_\epsilon)=\sum_{i=1}^n [f(t_i'')-f(t_i')](x_i-x_{i-1})$$
$$<\sum_{i=1}^n \frac{\epsilon}{b-a}(x_i-x_{i-1})$$
$$=\frac{\epsilon}{b-a}\cdot(b-a)$$
$$=\epsilon$$

$\because$ $t_i', t_i''\in[x_{i-1},x_i]$
$\therefore$ $|t_i''-t_i'|\le|x_i-x_{i-1}|\le\|\mathcal{P}\|<\delta$
$\therefore$ $|f(t_i'')-f(t_i')|<\frac{\epsilon}{b-a}$

Hence, Squeeze Thm implies: $f\in\mathcal{R}[a,b]$. #

Thm Let $f:[a,b]\to\mathbb{R}$ be monotone on $[a,b]$ with $-\infty<a<b<\infty$, then $f\in\mathcal{R}[a,b]$.

pf: Assume f is increasing (decreasing is similar).
Take the uniform partition $\mathcal{P}=\{[x_{i-1},x_i]\}_{i=1}^{n}$ of $[a,b]$:

$$x_i = a+\frac{b-a}{n}\cdot i,\quad i=0,1,\cdots,n$$

so that $x_i-x_{i-1}=\frac{b-a}{n}$, $i=1,\cdots,n$.

$\because f$ is increasing on $[a,b]$

$\therefore f(x_{i-1})\le f(x)\le f(x_i)$, $\forall x\in[x_{i-1},x_i]$; $i=1,2,\cdots,n$.

In terms of this, define two step functions:

$$\alpha_n(x)=\begin{cases} f(x_{i-1}), & x\in[x_{i-1},x_i),\quad i=1,\cdots,n-1\\ f(x_{n-1}), & x\in[x_{n-1},x_n],\end{cases}$$

and

$$w_n(x)=\begin{cases} f(x_i), & x\in[x_{i-1},x_i),\quad i=1,\cdots,n-1\\ f(x_n), & x\in[x_{n-1},x_n].\end{cases}$$

Obvious to see:

① $\alpha_n\in\mathcal{R}[a,b]$ and $w_n\in\mathcal{R}[a,b]$ satisfy

$$\alpha_n(x)\le f(x)\le w_n(x),\quad \forall x\in[a,b]$$

②
$$\int_a^b(w_n-\alpha_n)=\sum_{i=1}^{n}\left[f(x_i)-f(x_{i-1})\right](x_i-x_{i-1})$$

$$=\sum_{i=1}^{n}[f(x_i)-f(x_{i-1})]\cdot\frac{b-a}{n}$$

$$=\frac{b-a}{n}\sum_{i=1}^{n}[f(x_i)-f(x_{i-1})]$$

$$=\frac{\underbrace{(b-a)[f(b)-f(a)]}_{\text{an integer}}}{n}$$

Now, let $\epsilon>0$, then choose n_ϵ such that

$$n_\epsilon>\frac{(b-a)[f(b)-f(a)]}{\epsilon}$$

We have:

$\alpha_{n_\epsilon}, w_{n_\epsilon}\in\mathcal{R}[a,b]$ such that

$$\alpha_{n_\epsilon}(x)\le f(x)\le w_{n_\epsilon}(x),\quad\forall x\in[a,b]$$

$$\int_a^b(w_{n_\epsilon}-\alpha_{n_\epsilon})<\epsilon$$

Squeeze Thm implies: $f\in\mathcal{R}[a,b]$. #

Thm (Additivity Theorem)

Let $-\infty < a < c < b < \infty$,
then $f \in \mathcal{R}[a,b]$ iff $f|_{[a,c]} \in \mathcal{R}[a,c]$ & $f|_{[c,b]} \in \mathcal{R}[c,b]$.
In this case, $$\int_a^b f = \int_a^c f + \int_c^b f.$$

pf: "$\Rightarrow$" Let $\epsilon > 0$.

$\because f \in \mathcal{R}[a,b]$

$\therefore$ Cauchy Criterion tells:

$$(*)\left[\begin{array}{l} \exists \eta_\epsilon > 0 \text{ s.t. } \forall \dot{\mathcal{P}}, \dot{\mathcal{Q}} \text{ with } \|\dot{\mathcal{P}}\| < \eta_\epsilon \,\&\, \|\dot{\mathcal{Q}}\| < \eta_\epsilon, \\ \qquad |S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| < \epsilon \end{array}\right.$$

Set $f_1 = f|_{[a,c]} : [a,c] \to \mathbb{R}$.

Let $\dot{\mathcal{P}}_1, \dot{\mathcal{Q}}_1$ be any tagged partions of $[a,c]$ with $\|\dot{\mathcal{P}}_1\| < \eta_\epsilon$ & $\|\dot{\mathcal{Q}}_1\| < \eta_\epsilon$.

One can extend $\dot{\mathcal{P}}_1, \dot{\mathcal{Q}}_1$ to tagged partitions $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ of $[a,b]$ such that

- $\|\dot{\mathcal{P}}\| < \eta_\epsilon$, $\|\dot{\mathcal{Q}}\| < \eta_\epsilon$
- $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ have the same points and tags in $[c,b]$

We notice:

(Why?) $\longrightarrow$ $S(f_1;\dot{\mathcal{P}}_1) - S(f_1;\dot{\mathcal{Q}}_1) = S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})$

therefore, by $(*)$,

$$|S(f_1;\dot{\mathcal{P}}_1) - S(f_1;\dot{\mathcal{Q}}_1)| < \epsilon, \ \forall \dot{\mathcal{P}}_1, \dot{\mathcal{Q}}_1 \text{ with } \|\dot{\mathcal{P}}_1\| < \eta_\epsilon \,\&\, \|\dot{\mathcal{Q}}_1\| < \eta_\epsilon.$$

Cauchy Criterion shows that
$$f_1 = f|_{[a,c]} \in \mathcal{R}[a,c].$$

In the same way
$$f_2 = f|_{[c,b]} \in \mathcal{R}[c,b].$$

"$\Leftarrow$" Assume: $f_1 = f|_{[a,c]} \in \mathcal{R}[a,c]$ with $L_1 = \int_a^c f_1 = \int_a^c f \in \mathbb{R}$,
$f_2 = f|_{[c,b]} \in \mathcal{R}[c,b]$ with $L_2 = \int_c^b f_2 = \int_c^b f \in \mathbb{R}$.

Let $\epsilon > 0$.

$f_1 \in \mathcal{R}[a,c]$ gives: $\exists \delta' > 0$ s.t. $\forall$ tagged partition $\dot{\mathcal{P}}_1$ of $[a,c]$ with $\|\dot{\mathcal{P}}_1\| < \delta'$,

$$|S(f_1;\dot{P}_1) - L_1| < \epsilon/3.$$

Similarly, $f_2 \in \mathcal{R}[c,b]$ gives:

$\therefore$ $\exists \delta'' > 0$ s.t. $\forall$ tagged partition $\dot{P}_2$ of $[c,b]$ with $\|\dot{P}_2\| < \delta''$,

$$|S(f_2;\dot{P}_2) - L_2| < \epsilon/3.$$

Moreover, Boundedness Thm tells:

$f_1 = f|_{[a,c]}$ and $f_2 = f|_{[c,b]}$ are bounded,

$\therefore$ f is bounded on $[a,b]$,

i.e. $\exists M > 0$ s.t. $|f(x)| \le M$, $\forall x \in [a,b]$.

Now, define

$$\delta_\epsilon = \min\{\delta', \delta'', \tfrac{\epsilon}{6M}\} > 0.$$

$\boxed{\text{Claim}}$: $\forall$ tagged partition $\dot{Q}$ of $[a,b]$ with $\|\dot{Q}\| < \delta_\epsilon$,

$$|S(f;\dot{Q}) - (L_1 + L_2)| < \epsilon.$$

If the Claim holds, then $f \in \mathcal{R}[a,b]$ with

$$\int_a^b f = L_1 + L_2 = \int_a^c f + \int_c^b f,$$

and the proof is done.

<u>Proof of Claim:</u>

Let $\dot{Q} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ with $\|\dot{Q}\| < \delta_\epsilon$.

<u>Case (i)</u>: $c = x_k$ for some $k \in \{1, \cdots, n-1\}$ ($x_0 = a$ and $x_n = b$ are exclusive)

In this case,

$\dot{Q}_1 = \{[x_{i-1}, x_i], t_i\}_{i=1}^k$ is a tagged partition of $[a,c]$ with $\|\dot{Q}_1\| \le \|\dot{Q}\| < \delta_\epsilon < \delta'$,

$\dot{Q}_2 = \{[x_{i-1}, x_i], t_i\}_{i=k+1}^n$ is a tagged partition of $[c,b]$ with $\|\dot{Q}_2\| \le \|\dot{Q}\| < \delta_\epsilon < \delta''$,

and

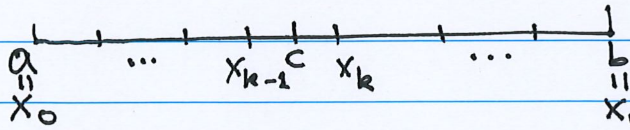
$$\dot{Q} = \dot{Q}_1 \cup \dot{Q}_2 \quad \text{with } S(f;\dot{Q}) = S(f_1;\dot{Q}_1) + S(f_2;\dot{Q}_2)$$

therefore,

$$|S(f;\dot{Q}) - (L_1 + L_2)|$$
$$\le |S(f_1;\dot{Q}_1) - L_1| + |S(f_2;\dot{Q}_2) - L_2|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2}{3}\epsilon < \epsilon .$$

<u>Case (ii)</u>: $c \in (x_{k-1}, x_k)$ for some $k \in \{1, \cdots, n\}$.



Define

$$\dot{Q}_1 = \{[x_{i-1}, x_i], t_i\}_{i=1}^{k-1} \cup \{[x_{k-1}, c], c\}$$

a tagged partition of $[a, c]$

$$\dot{Q}_2 = \{[c, x_k], c\} \cup \{[x_{i-1}, x_i], t_i\}_{i=k+1}^{n}$$

a tagged partition of $[c, b]$

<u>Compute</u>:

$$S(f; \dot{Q})$$
$$= \sum_{i=1}^{k-1} f(t_i)(x_i - x_{i-1}) + f(t_k)(x_k - x_{k-1}) + \sum_{i=k+1}^{n} f(t_i)(x_i - x_{i-1})$$
$$= \left[\sum_{i=1}^{k-1} f(t_i)(x_i - x_{i-1}) + f(c)(c - x_{k-1})\right] - f(c)(c - x_{k-1})$$
$$+ f(t_k)(x_k - x_{k-1})$$
$$+ \left[f(c)(x_k - c) + \sum_{i=k+1}^{n} f(t_i)(x_i - x_{i-1})\right] - f(c)(x_k - c)$$

$$= S(f_1; \dot{Q}_1) - f(c)(c - x_{k-1}) + f(t_k).(x_k - x_{k-1})$$
$$+ S(f_2; \dot{Q}_2) - f(c)(x_k - c)$$

implying

$$|S(f; \dot{Q}) - S(f_1; \dot{Q}_1) - S(f_2; \dot{Q}_2)|$$
$$\leq |f(t_k) - f(c)| \cdot (x_k - x_{k-1})$$
$$\leq 2M \cdot \|\dot{Q}\|$$
$$< 2M \cdot \frac{\epsilon}{6M} = \frac{\epsilon}{3}$$

Note: $\|\dot{Q}_1\| \leq \|\dot{Q}\|$ $\left(\because 0 < \underbrace{c - x_{k-1}}_{\text{length of the last subinterval of } \dot{Q}_1} < x_k - x_{k-1}\right)$

then $\|\dot{Q}_1\| < \delta_\epsilon < \delta'$

$$\therefore |S(f_1; \dot{Q}_1) - L_1| < \epsilon/3 .$$

Similarly, $\|\dot{Q}_2\| \leq \|\dot{Q}\| < \delta_\epsilon < \delta''$

$$\therefore |S(f_2; \dot{Q}_2) - L_2| < \epsilon/3 .$$

Collecting those estimates,

$$
\begin{aligned}
|S(f;\dot{Q}) - (L_1+L_2)| &\le |S(f;\dot{Q}) - S(f_1;\dot{Q}_1) - S(f_2;\dot{Q}_2)| \\
&\quad + |S(f_1;\dot{Q}_1) - L_1| \\
&\quad + |S(f_2;\dot{Q}_2) - L_2| \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon
\end{aligned}
$$

this completes the proof of Claim and the proof of Thm.

##

Cor 1: Let $f \in \mathcal{R}[a,b]$ and $[c,d] \subset [a,b]$, then $f \in \mathcal{R}[c,d]$.

Pf: Additivity Thm tells:

$$f \in \mathcal{R}[a,b] \Rightarrow f \in \mathcal{R}[c,b] \Rightarrow f \in \mathcal{R}[c,d]. \quad \#$$

Cor 2: Let $f \in \mathcal{R}[a,b]$ and $a = c_0 < c_1 < \cdots < c_m = b$, then $f|_{[c_{i-1}, c_i]} \in \mathcal{R}[c_{i-1}, c_i]$ with

$$\int_a^b f = \sum_{i=1}^m \int_{c_{i-1}}^{c_i} f .$$

Pf. By induction. omitted. #

Def: Let $f \in \mathcal{R}[a,b]$ and $\alpha, \beta \in [a,b]$ with $\alpha < \beta$, define

$$\int_\beta^\alpha f \stackrel{\text{def.}}{=} -\int_\alpha^\beta f \quad \text{and} \quad \int_\alpha^\alpha f = 0 .$$

Thm: Let $f \in \mathcal{R}[a,b]$ and $\alpha, \beta, \gamma \in [a,b]$, then

$$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f \quad \text{——} \; (*)$$

in the sense that the existence of any two of three integrals implies the existence of the third one and $(*)$ holds.

Pf: First observe:

"the existence of any two of those integrals

$\Longrightarrow$ the existence of the 3rd one"

Case (i): any two of α, β, γ are equal.

then $(*)$ trivially holds. (check it!)

Case (ii) Let $\alpha, \beta, \gamma \in [a, b]$ be distinct.

Consider

$$L(\alpha, \beta, \gamma) \overset{\text{def.}}{=} \int_\alpha^\beta f + \int_\beta^\gamma f + \int_\gamma^\alpha f$$

$$= \int_\alpha^\beta f - \int_\gamma^\beta f - \int_\alpha^\gamma f \qquad \text{———} (\star)$$

it's direct to verify

$$L(\alpha, \beta, \gamma) = L(\beta, \gamma, \alpha) = L(\gamma, \alpha, \beta)$$

$$= -L(\alpha, \gamma, \beta) = -L(\gamma, \beta, \alpha) = -L(\beta, \alpha, \gamma).$$

If $\alpha < \gamma < \beta$, then Additivity Thm gives

$$L(\alpha, \beta, \gamma) = \int_\alpha^\beta f - \left(\int_\alpha^\gamma f + \int_\gamma^\beta f \right) = 0$$

Similarly, in all other situations

$\gamma < \beta < \alpha$, $\beta < \alpha < \gamma$

$\gamma < \alpha < \beta$, $\alpha < \beta < \gamma$, & $\beta < \gamma < \alpha$,

we still have

$$L(\alpha, \beta, \gamma) = 0.$$

Therefore, $\forall \alpha, \beta, \gamma \in [a, b]$, it holds

$$L(\alpha, \beta, \gamma) = 0$$

i.e. by $(\star)$,

$$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f. \qquad \#$$