Topic #5 Riemann Integral

Def. Let $I=[a,b]$, $-\infty<a<b<\infty$

then a partition of I is a finite, ordered set:

$$\mathcal{P}=(x_0, x_1, \cdots, x_n)$$

where x_i $(0\le i\le n)$ are points in I such that

$$a=x_0<x_1<\cdots<x_n=b.$$

Note: A partition $\mathcal{P}=(x_0, x_1, \cdots, x_n)$ divides I into subintervals

$$I_1=[x_0,x_1],\ I_2=[x_1,x_2],\ \cdots,\ I_n=[x_{n-1},x_n]$$

with interiors non-overlapping.

As such, we also denote $\mathcal{P}$ as

$$\mathcal{P}=\{[x_{i-1}, x_i]\}_{i=1}^{n}$$

Def: The norm (or mesh) of a partion $\mathcal{P}=\{[x_{i-1},x_i]\}_{i=1}^{n}$ is defined as

$$\|\mathcal{P}\|=\max_{i=1,\cdots,n}\{x_i-x_{i-1}\}$$

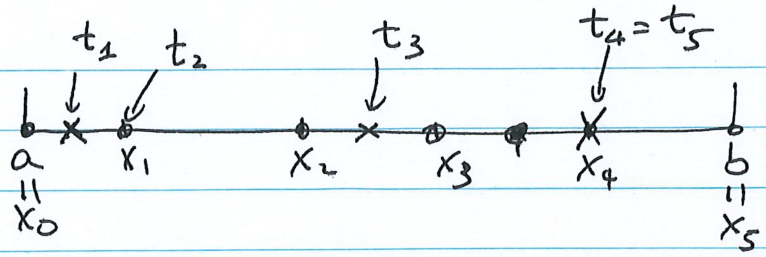
$$=\max\{x_1-x_0,\ x_2-x_1,\ \cdots,\ x_n-x_{n-1}\}$$

i.e. length of the largest subinterval of $\mathcal{P}$.

Def. (1) Let $t_i\in I_i:=[x_{i-1},x_i]$ $(i=1,\cdots,n)$ be chosen in each subinterval of a partition $\mathcal{P}=\{[x_{i-1},x_i]\}_{i=1}^{n}$ of $I=[a,b]$, then t_i $(1\le i\le n)$ are called tags of I_i.

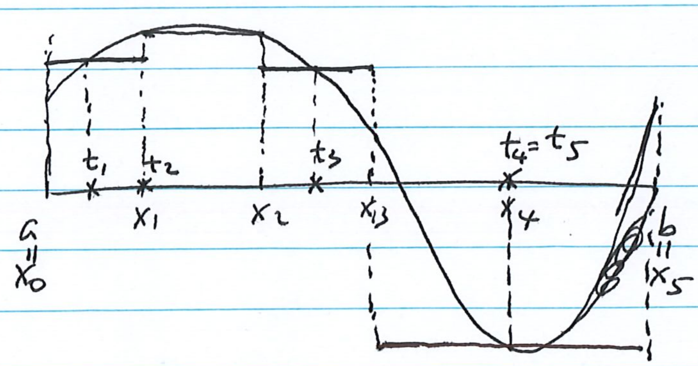
(2) The partition $\mathcal{P}=\{[x_{i-1},x_i]\}_{i=1}^{n}$, together with tags t_i is called a tagged partition of $I=[a,b]$, denoted as

$$\dot{\mathcal{P}}=\{[x_{i-1},x_i],\ t_i\}_{i=1}^{n}.$$



Def. Let $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ be a tagged partition of $I = [a,b]$, then the <u>Riemann sum</u> of a function $f : [a,b] \to \mathbb{R}$ is defined by

$$S(f; \dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)\,(x_i - x_{i-1})$$



$S(f; \dot{\mathcal{P}})$ = sum of signed areas of those n rectangles with bases $[x_{i-1}, x_i]$ and heights $f(t_i)$, $i = 1, 2, \cdots, n$.

Def. (1) A function $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ if $\exists L \in \mathbb{R}$ s.t. $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\forall$ tagged partition $\dot{\mathcal{P}}$ of $[a,b]$ with $\|\dot{\mathcal{P}}\| < \delta$,
$$|S(f; \dot{\mathcal{P}}) - L| < \epsilon.$$

(2) $\mathcal{R}[a,b] = \{ f : [a,b] \to \mathbb{R} : f \text{ is Riemann integrable on } [a,b] \}$

(3) If $f \in \mathcal{R}[a,b]$, then such $L \in \mathbb{R}$ is unique, called the Riemann integral of f on $[a,b]$ and denoted by
$$L = \int_a^b f \quad \text{or} \quad \int_a^b f(x)\,dx$$
(x is dummy) #

Remark: Sometimes one writes $L = \text{"}\lim_{\|\dot{\mathcal{P}}\| \to 0}\text{"} S(f; \dot{\mathcal{P}})$ regarding

L as "the limit" of $S(f; \dot{\mathcal{P}})$ as $\|\dot{\mathcal{P}}\| \to 0$. However, $S(f; \dot{\mathcal{P}})$ is not a function of $\|\dot{\mathcal{P}}\|$ (the same $\|\dot{\mathcal{P}}\|$ may correspond to many different $\dot{\mathcal{P}}$'s), so such limit is not the limit defined in the classical sense.

Thm. Let $f \in \mathcal{R}[a,b]$, then $L = \int_a^b f$ is uniquely determined.

Pf. Suppose: $L_1, L_2 \in \mathbb{R}$ both satisfy the def.
Let $\epsilon > 0$, then

$\exists \delta_1 > 0$ s.t. $|S(f; \dot{\mathcal{P}}_1) - L_1| < \frac{\epsilon}{2}$ $\forall \dot{\mathcal{P}}_1$ with $\|\dot{\mathcal{P}}_1\| < \delta_1$

$\exists \delta_2 > 0$ s.t. $|S(f; \dot{\mathcal{P}}_2) - L_2| < \frac{\epsilon}{2}$ $\forall \dot{\mathcal{P}}_2$ with $\|\dot{\mathcal{P}}_2\| < \delta_2$

Define $\delta = \min\{\delta_1, \delta_2\}$, then $\delta > 0$.
Let $\dot{\mathcal{P}}$ be a tagged partition with $\|\dot{\mathcal{P}}\| < \delta$, then

$$\|\dot{\mathcal{P}}\| < \delta_1 \text{ and } \|\dot{\mathcal{P}}\| < \delta_2.$$

Hence $|S(f; \dot{\mathcal{P}}) - L_1| < \epsilon/2$ and $|S(f; \dot{\mathcal{P}}) - L_2| < \epsilon/2$.
This gives

$$\begin{aligned} |L_1 - L_2| &\leq |L_1 - S(f; \dot{\mathcal{P}})| + |S(f; \dot{\mathcal{P}}) - L_2| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $|L_1 - L_2| = 0$, i.e. $L_1 = L_2$.

#

Thm Let $f(x) = g(x)$ except for a finite number of points of $[a,b]$ with $g \in \mathcal{R}[a,b]$, then $f \in \mathcal{R}[a,b]$ and $\int_a^b f = \int_a^b g$.

#

Pf Only prove the case $f(x)=g(x)$ except for one point in $[a,b]$; the general case can be treated by induction.

Assume: $\exists\, c\in[a,b]$ s.t. $f(c)\neq g(c)$ and $f(x)=g(x)$, $\forall x\in[a,b]\setminus\{c\}$

Since $g\in \mathcal{R}[a,b]$, $\exists\, L\in\mathbb{R}$ s.t. $L=\int_a^b g$.

Let $\dot{\mathcal{P}}=\{[x_{i-1},x_i],t_i\}_{i=1}^{n}$ be a tagged partion of $[a,b]$, then

either (i) $c\in(x_{i_0-1},x_{i_0})$ for some $i_0\in\{1,2,\cdots,n\}$

or (ii) $c=x_{i_0}$ for some $i_0\in\{1,2,\cdots,n\}$.

(at most two subintervals contain c)

<u>In case (i)</u>: $f(x)=g(x)$, $\forall x\in[x_{i-1},x_i]$ with $i\neq i_0$

then $f(t_i)=g(t_i)$ for $i\neq i_0$ and

~~and~~

$$
\begin{aligned}
&S(f;\dot{\mathcal{P}})-S(g;\dot{\mathcal{P}})\\
&=\sum_{i=1}^{n} f(t_i)(x_i-x_{i-1})-\sum_{i=1}^{n} g(t_i)(x_i-x_{i-1})\\
&=\sum_{i=1}^{n}[f(t_i)-g(t_i)](x_i-x_{i-1})\\
&=\underbrace{\sum_{\substack{i=1\\ i\neq i_0}}^{n}[f(t_i)-g(t_i)](x_i-x_{i-1})}_{=0}+[f(t_{i_0})-g(t_{i_0})][x_{i_0}-x_{i_0-1}]\\
&=[f(t_{i_0})-g(t_{i_0})](x_{i_0}-x_{i_0-1})
\end{aligned}
$$

implying

$$
\begin{aligned}
|S(f;\dot{\mathcal{P}})-S(g;\dot{\mathcal{P}})| &\le |f(t_{i_0})-g(t_{i_0})|\cdot|x_{i_0}-x_{i_0-1}|\\
&\le (|f(c)|+|g(c)|)\,\|\dot{\mathcal{P}}\|
\end{aligned}
$$

($\because$ either $t_{i_0}=c$ or $t_{i_0}\neq c$)

<u>In case (ii)</u>: $c=x_{i_0}\in[x_{i_0-1},x_{i_0}]\cap[x_{i_0},x_{i_0+1}]$

$\therefore$ $f(x)=g(x)$, $\forall x\in[x_{i-1},x_i]$ with $i\neq i_0,\ i_0+1$.

Similarly

$$
\begin{aligned}
&S(f;\dot{\mathcal{P}})-S(g;\dot{\mathcal{P}})\\
&=\underbrace{\sum_{\substack{i=1\\ i\neq i_0,\,i_0+1}}^{n}[f(t_i)-g(t_i)](x_i-x_{i-1})}_{=0}+[f(t_{i_0})-g(t_{i_0})][x_{i_0}-x_{i_0-1}]\\
&\quad+[f(t_{i_0+1})-g(t_{i_0+1})][x_{i_0+1}-x_{i_0}]
\end{aligned}
$$

implying

$$|S(f;\dot{p})-S(g;\dot{p})|$$
$$\le |f(t_{i_0})-g(t_{i_0})|\cdot\|\dot{p}\| + |f(t_{i_0+1})-g(t_{i_0+1})|\cdot\|\dot{p}\|$$
$$\le 2(|f(c)|+|g(c)|)\|\dot{p}\|$$

Thus, in both cases,

$$|S(f;\dot{p})-S(g;\dot{p})| \le 2(|f(c)|+|g(c)|)\|\dot{p}\|.$$

Let $\epsilon>0$. Define $\delta_1 = \dfrac{\epsilon}{5(|f(c)|+|g(c)|)+1}$, then $\delta_1>0$.
And, $\forall \dot{p}$ with $\|\dot{p}\|<\delta_1$,

$$|S(f;\dot{p})-S(g;\dot{p})| \le 2(|f(c)|+|g(c)|)\|\dot{p}\|$$
$$< 2(|f(c)|+|g(c)|)\cdot\frac{\epsilon}{5(|f(c)|+|g(c)|)+1}$$
$$< \frac{2}{5}\epsilon < \frac{\epsilon}{2}$$

Moreover, $g\in\mathcal{R}[a,b]$, ~~tells: $\exists\delta$~~ let $L=\int_a^b g\in\mathbb{R}$, then $\exists\,\delta_2>0$ s.t.

$$|S(g;\dot{p})-L|<\epsilon/2 \quad \forall \dot{p} \text{ with } \|\dot{p}\|<\delta_2.$$

Define $\delta=\min\{\delta_1,\delta_2\}>0$, then $\forall \dot{p}$ with $\|\dot{p}\|<\delta$,

$$|S(f;\dot{p})-L| \le |S(f;\dot{p})-S(g;\dot{p})| + |S(g;\dot{p})-L|$$
$$< \epsilon/2+\epsilon/2 = \epsilon.$$

$\therefore f\in\mathcal{R}[a,b]$ and $\int_a^b f = L = \int_a^b g$. ##

Examples:

(a) Let $f \equiv k$ be a constant function, then $f \in \mathcal{R}[a,b]$.

Pf: Let $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ be a tagged partition of $[a,b]$, then

$$\begin{aligned} S(f;\dot{\mathcal{P}}) &= \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n k\,(x_i - x_{i-1}) \\ &= k \sum_{i=1}^n (x_i - x_{i-1}) \\ &= k\,(x_n - x_0) \\ &= k\,(b-a). \end{aligned}$$

Let $\epsilon > 0$, then one can pick any $\delta > 0$ and have

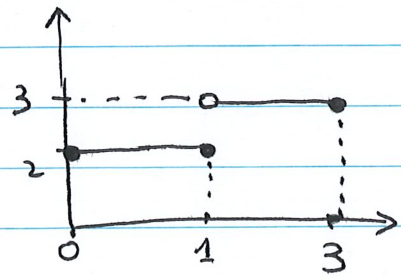
$$|S(f;\dot{\mathcal{P}}) - k(b-a)| = 0 < \epsilon, \ \forall \dot{\mathcal{P}} \text{ with } \|\dot{\mathcal{P}}\| < \delta.$$

$\therefore$ $f \equiv k \in \mathcal{R}[a,b]$

and $\int_a^b k = k(b-a)$.

(b) Let $g : [0,3] \to \mathbb{R}$ be defined as

$$g(x) = \begin{cases} 3, & 1 < x \leq 3 \\ 2, & 0 \leq x \leq 1 \end{cases}$$



then $g \in \mathcal{R}[0,3]$ and $\int_0^3 g = 8$

Pf: Let $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ be a tagged partition of $[0,3]$.

There is $k \in \{1, \cdots, n\}$ such that $t_k \leq 1 < t_{k+1}$ (think about all cases)

Denote $\dot{\mathcal{P}}_1 = \{[x_{i-1}, x_i], t_i\}_{i=1}^k$: tagged partition of $[0, x_k]$

and $\dot{\mathcal{P}}_2 = \{[x_{i-1}, x_i], t_i\}_{i=k+1}^n$: tagged partition of $[x_k, 3]$

We may write

$$S(g;\dot{P}) = \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1})$$
$$= \sum_{i=1}^{k} g(t_i)(x_i - x_{i-1}) + \sum_{i=k+1}^{n} g(t_i)(x_i - x_{i-1})$$
$$= S(g;\dot{P}_1) + S(g;\dot{P}_2),$$

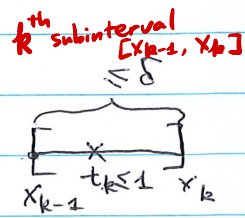
where

$$S(g;\dot{P}_1) = \sum_{i=1}^{k} g(t_i)(x_i - x_{i-1})$$
$$= \sum_{i=1}^{k} 2(x_i - x_{i-1}) \qquad \left(\because\ t_k \le 1 \quad \therefore\ 0 \le t_1 \le \cdots \le t_k \le 1\right)$$
$$= 2(x_k - x_0)$$
$$= 2(x_k - 0)$$
$$= 2x_k$$

and

$$S(g;\dot{P}_2) = \sum_{i=k+1}^{n} g(t_i)(x_i - x_{i-1})$$
$$= \sum_{i=k+1}^{n} 3(x_i - x_{i-1}) \qquad \left(\because\ 1 < t_{k+1} \quad \therefore\ 1 < t_{k+1} \le \cdots \le t_n \le 3\right)$$
$$= 3\sum_{i=k+1}^{n}(x_i - x_{i-1})$$
$$= 3(x_n - x_k)$$
$$= 3(3 - x_k).$$

Claim: Let ~~$\|\dot{P}\|$~~ $\delta > 0$ be such that $\|\dot{P}\| < \delta$, then $1 - \delta < x_k \le 1 + \delta$

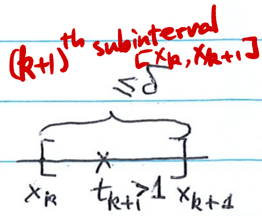


In fact, by the fact that

$$t_k \le 1, \quad x_{k-1} \le t_k \le x_k, \quad x_k - x_{k-1} \le \delta,$$

it holds:

$$x_k \le x_{k-1} + \delta \le t_k + \delta \le 1 + \delta.$$



By the fact that

$$t_{k+1} > 1, \quad x_k \le t_{k+1} \le x_{k+1}, \quad x_{k+1} - x_k \le \delta$$

it holds:

$$x_k \ge x_{k+1} - \delta \ge t_{k+1} - \delta > 1 - \delta. \quad \#$$

By the claim,

$$2(1-\delta) < S(g;\dot{\mathcal{P}}_1) = 2x_k \leq 2(1+\delta)$$

$$3(2-\delta) = 3(3-(1+\delta)) < S(g;\dot{\mathcal{P}}_2) = 3(3-x_k) \leq 3(3-(1-\delta)) = 3(2+\delta)$$

Thus, for $\dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\|<\delta$,

$$2(1-\delta)+3(2-\delta) \leq S(g;\dot{\mathcal{P}}) \leq 2(1+\delta)+3(2+\delta)$$

i.e.

$$8-5\delta \leq S(g;\dot{\mathcal{P}}) \leq 8+5\delta$$

$$\therefore \quad |S(g;\dot{\mathcal{P}})-8| \leq 5\delta.$$

Let $\delta = \frac{\epsilon}{10} > 0$, then

$$|S(g;\dot{\mathcal{P}})-8| \leq 5\cdot\frac{\epsilon}{10} < \epsilon, \quad \forall \dot{\mathcal{P}} \text{ with } \|\dot{\mathcal{P}}\|<\delta,$$

Therefore

$$g \in \mathcal{R}[0,3] \text{ with } \int_0^3 g = 8. \qquad \#\#$$

(c) Let $h(x)=x$, $0\leq x\leq 1$, then $h\in\mathcal{R}[0,1]$ with $\int_0^1 h = \frac{1}{2}$.

Pf: Let $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of $I=[0,1]$.
Take tags $\xi_i := \frac{x_{i-1}+x_i}{2}$ to be the mid-points of $[x_{i-1},x_i]$.
For the tagged partion $\dot{\mathcal{Q}} = \{[x_{i-1},x_i];\, \xi_i\}_{i=1}^n$,

$$\begin{aligned}
S(h;\dot{\mathcal{Q}}) &= \sum_{i=1}^n h(\xi_i)(x_i-x_{i-1}) \\
&= \sum_{i=1}^n \xi_i (x_i-x_{i-1}) \\
&= \sum_{i=1}^n \frac{x_i+x_{i-1}}{2}\cdot(x_i-x_{i-1}) \\
&= \frac{1}{2}\sum_{i=1}^n (x_i^2-x_{i-1}^2) \\
&= \frac{1}{2}(x_n^2-x_0^2) \\
&= \frac{1}{2}(1^2-0^2) \\
&= \frac{1}{2}.
\end{aligned}$$

Let $\delta>0$ be such that $\dot{P}=\{[x_{i-1},x_i],t_i\}_{i=1}^{n}$ is a tagged partition with the same partion but arbitrary tags t_i such that $\|\dot{P}\|<\delta$.
As such,

$$
\begin{aligned}
&|S(h;\dot{P})-S(h;\dot{Q})|\\
&=\left|\sum_{i=1}^{n}h(t_i)(x_i-x_{i-1})-\sum_{i=1}^{n}h(q_i)(x_i-x_{i-1})\right|\\
&=\left|\sum_{i=1}^{n}t_i(x_i-x_{i-1})-\sum_{i=1}^{n}q_i(x_i-x_{i-1})\right|\\
&=\left|\sum_{i=1}^{n}(t_i-q_i)(x_i-x_{i-1})\right|\\
&\leq\sum_{i=1}^{n}|t_i-q_i|\cdot(x_i-x_{i-1})\\
&<\delta\sum_{i=1}^{n}(x_i-x_{i-1}) \qquad \left[\begin{array}{l}\because\ t_i,q_i\in[x_{i-1},x_i],\quad x_i-x_{i-1}<\delta\\ \therefore\ |t_i-q_i|\leq x_i-x_{i-1}<\delta\end{array}\right]\\
&=\delta(x_n-x_0)\\
&=\delta(1-0)=\delta.
\end{aligned}
$$

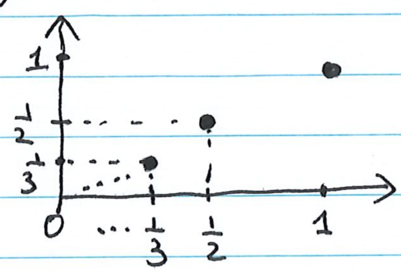
Now, take $\delta=\epsilon>0$. $\forall\,\dot{P}$ with $\|\dot{P}\|<\delta$,

$$|S(h;\dot{P})-\tfrac{1}{2}|=|S(h;\dot{P})-S(h;\dot{Q})|<\epsilon$$

$\therefore$ $h\in\mathcal{R}[0,1]$ with $\int_0^1 h=\frac{1}{2}$. ##

— End of Feb 5 —

(d) Let $G(x)=\begin{cases}\frac{1}{n}, & \text{if } x=\frac{1}{n}\ (n=1,2,\cdots)\\ 0 & \text{elsewhere on } [0,1]\end{cases}$



then $G\in\mathcal{R}[0,1]$ with $\int_0^1 G=0$.

Pf: Let $\epsilon>0$. Denote

$$
\begin{aligned}
E_\epsilon&=\{x\in[0,1]:\ G(x)\geq\epsilon\}\\
&=\{1,\tfrac{1}{2},\cdots,\tfrac{1}{N_\epsilon}\}
\end{aligned}
$$

where $N_\epsilon=[\frac{1}{\epsilon}]$ is the largest integer $\leq\frac{1}{\epsilon}$

Note: # of $E_\epsilon=N_\epsilon$. Take $\delta=\frac{\epsilon}{2N_\epsilon}>0$.

Let $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ be a tagged partition with $\|\dot{\mathcal{P}}\| < \delta$, then

$$S(G;\dot{\mathcal{P}}) = \sum_{i=1}^{n} G(t_i)(x_i - x_{i-1})$$

$$= \underbrace{\sum_{\substack{i=1 \\ t_i \notin E_\epsilon}}^{n} G(t_i)(x_i - x_{i-1})}_{(\mathrm{I})} + \underbrace{\sum_{\substack{i=1 \\ t_i \in E_\epsilon}}^{n} G(t_i)(x_i - x_{i-1})}_{(\mathrm{II})}$$

part (I): $t_i \notin E_\epsilon \Rightarrow 0 \leq G(t_i) < \epsilon$,

$\therefore\ 0 \leq (\mathrm{I}) < \epsilon \sum_{i=1}^{n} (x_i - x_{i-1}) = \epsilon(1-0) = \epsilon$.

part (II): $\because\ 0 \leq G(x) \leq 1,\ 0 \leq x \leq 1$; # of $E_\epsilon = N_\epsilon$

$$\therefore\ 0 \leq (\mathrm{II}) \leq \sum_{\substack{i=1 \\ t_i \in E_\epsilon}}^{n} \delta$$

each $t_i \in E_\epsilon$ may be counted twice

$$\leq 2N_\epsilon \cdot \delta = \epsilon$$

Hence,

$$0 \leq S(G,\dot{\mathcal{P}}) = (\mathrm{I}) + (\mathrm{II}) < \epsilon + \epsilon = 2\epsilon,\ \forall \dot{\mathcal{P}} \text{ with } \|\dot{\mathcal{P}}\| < \delta.$$

therefore,

$$G \in \mathcal{R}[0,1] \text{ with } \int_0^1 G = 0. \quad \#$$

Properties of Integrals

Thm: Let $f, g \in \mathcal{R}[a,b]$, then

(a) $kf \in \mathcal{R}[a,b]$, $\forall k \in \mathbb{R}$ with $\int_a^b kf = k\int_a^b f$

(b) $f+g \in \mathcal{R}[a,b]$ with $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

(c) If $f(x) \leq g(x)$, $\forall x \in [a,b]$, then $\int_a^b f \leq \int_a^b g$

Pf (a) Omitted, similar to the proof of (b).

(b) Let $\epsilon > 0$. Since $f, g \in \mathcal{R}[a,b]$,

$$\exists \delta_1 > 0 \text{ s.t. } \left| S(f;\dot{\mathcal{P}}) - \int_a^b f \right| < \epsilon, \ \forall \dot{\mathcal{P}} \text{ with } \|\dot{\mathcal{P}}\| < \delta_1$$

and

$$\exists \delta_2 > 0 \text{ s.t. } \left| S(f;\dot{\mathcal{P}}) - \int_a^b g \right| < \epsilon, \ \forall \dot{\mathcal{P}} \text{ with } \|\dot{\mathcal{P}}\| < \delta_2 .$$

Consider a tagged partition $\dot{\mathcal{P}} = \{ [x_{i-1}, x_i], t_i \}_{i=1}^n$ of $[a,b]$.

We have

$$\begin{aligned} S(f+g;\dot{\mathcal{P}}) &= \sum_{i=1}^n (f+g)(t_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) + \sum_{i=1}^n g(t_i)(x_i - x_{i-1}) \\ &= S(f;\dot{\mathcal{P}}) + S(g;\dot{\mathcal{P}}) \end{aligned}$$

Then, $\forall \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta := \min\{\delta_1, \delta_2\} > 0$,

$$\begin{aligned} &\left| S(f+g;\dot{\mathcal{P}}) - \left(\int_a^b f + \int_a^b g \right) \right| \\ &= \left| \left[S(f;\dot{\mathcal{P}}) - \int_a^b f \right] + \left[S(g;\dot{\mathcal{P}}) - \int_a^b g \right] \right| \\ &\le \left| S(f;\dot{\mathcal{P}}) - \int_a^b f \right| + \left| S(g;\dot{\mathcal{P}}) - \int_a^b g \right| \\ &< \epsilon + \epsilon \\ &= 2\epsilon . \end{aligned}$$

This proved that $f+g \in \mathcal{R}[a,b]$ with $\int_a^b (f+g) = \int_a^b f + \int_a^b g$. #

(c) Let $\epsilon > 0$. As in (b), $\exists \delta > 0$ s.t.

$$\left. \begin{aligned} \left| S(f;\dot{\mathcal{P}}) - \int_a^b f \right| < \epsilon \\ \left| S(g;\dot{\mathcal{P}}) - \int_a^b g \right| < \epsilon \end{aligned} \right\} \ \forall \dot{\mathcal{P}} \text{ with } \|\dot{\mathcal{P}}\| < \delta .$$

Consider $\dot{\mathcal{P}} = \{ [x_{i-1}, x_i], t_i \}_{i=1}^n$ of $[a,b]$.

Note:

$$\begin{aligned} S(f;\dot{\mathcal{P}}) &= \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) \\ &\le \sum_{i=1}^n g(t_i)(x_i - x_{i-1}) \quad \left(\because f(x) \le g(x), \forall x \in [a,b], \ x_i - x_{i-1} \ge 0 \right) \\ &= S(g;\dot{\mathcal{P}}) \end{aligned}$$

Therefore, $\int_a^b f - \epsilon < S(f;\dot{\mathcal{P}}) \le S(g;\dot{\mathcal{P}}) < \int_a^b g + \epsilon$

i.e. $\int_a^b f < \int_a^b g + 2\epsilon$.

Since $\epsilon > 0$ is arbitrary, it holds: $\int_a^b f \le \int_a^b g$. #

<u>Bounded Theorem:</u>

A Riemann integral function must be bounded.

<u>Thm</u>: Let $f \in \mathcal{R}[a,b]$, then f is bounded on $[a,b]$.

<u>Pf</u>: Otherwise f is unbounded on $[a,b]$.

Let $\int_a^b f = L \in \mathbb{R}$, then $\exists\, \delta > 0$ s.t.

$$|S(f;\dot{\mathcal{P}}) - L| < 1 \quad \forall \dot{\mathcal{P}} \text{ with } \|\dot{\mathcal{P}}\| < \delta.$$

This implies: $\forall\, \dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| < \delta$,

$$|S(f;\dot{\mathcal{P}})| \le |S(f;\dot{\mathcal{P}}) - L| + |L| < |L| + 1. \quad \text{—— } (*)$$

Let $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of $[a,b]$ with $\|\mathcal{P}\| < \delta$.

$\because f$ is unbounded on $[a,b]$,

$\therefore \exists$ a subinterval $[x_{i_0-1}, x_{i_0}]$ s.t.

f is unbounded on $[x_{i_0-1}, x_{i_0}]$

WLG, we also assume f is bounded on all other subintervals

Then, one can find $t_{i_0} \in (x_{i_0-1}, x_{i_0})$ s.t.

$$|f(t_{i_0})(x_{i_0} - x_{i_0-1})| > |L| + 1 + \left|\sum_{i \ne i_0} f(x_i)(x_i - x_{i-1})\right| \quad \text{—— } (**)$$

Now, we choose tags

$$t_i = \begin{cases} x_i & \text{if } i \ne i_0 \\ t_{i_0} & \text{if } i = i_0 \end{cases}$$

and for the tagged partition $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$,

$$S(f;\dot{\mathcal{P}}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$$

$$= f(t_{i_0})(x_{i_0} - x_{i_0-1}) + \sum_{i \ne i_0} f(x_i)(x_i - x_{i-1})$$

$$\Rightarrow f(t_{i_0})(x_{i_0} - x_{i_0-1}) = S(f;\dot{\mathcal{P}}) - \sum_{i \ne i_0} f(x_i)(x_i - x_{i-1})$$

$$\Rightarrow |f(t_{i_0})(x_{i_0} - x_{i_0-1})| \le |S(f;\dot{\mathcal{P}})| + \left|\sum_{i \ne i_0} f(x_i)(x_i - x_{i-1})\right|$$

$$< |L| + 1 + \left|\sum_{i \ne i_0} f(x_i)(x_i - x_{i-1})\right| \quad (\text{by } (*))$$

this contradicts $(**)$.

$\therefore f$ is bounded on $[a,b]$. #

Example: $\exists$ a function that is discontinuous at every rational number but is Riemann integrable.

In fact, define $h: [0,1] \to \mathbb{R}$ by

$$h(x) = \begin{cases} 0, & \text{if } x \text{ is irrational}, \in [0,1] \\ 1, & \text{if } x = 0 \\ \frac{1}{n}, & \text{if } x = \frac{m}{n} \in [0,1], \text{ where } m, n \in \mathbb{N} = \{1, 2, \cdots\} \text{ and g.c.d.}(m,n) = 1 \\ & \quad (\text{rational}) \end{cases}$$

(Thomae's function)

Show that $h \in \mathcal{R}[0,1]$ with $\int_0^1 h = 0$

RK: Recall (Chapter 5.1.6 (h) of the textbook) that h is discontinuous at every rational number in $[0,1]$ and continuous at every irrational number in $[0,1]$.

Pf: Let $\epsilon > 0$. Define $E_\epsilon = \{x \in [0,1] : h(x) \geq \frac{\epsilon}{2}\}$.

Note that E_ϵ is a finite set,

for instance, $\frac{\epsilon}{2} = \frac{1}{5}$, then

$$E_\epsilon = \{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}\}$$

Denote $N_\epsilon = \#$ of E_ϵ

Define $\delta_\epsilon = \frac{\epsilon}{4N_\epsilon} > 0$.

Then, $\forall \dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ with $\|\dot{\mathcal{P}}\| < \delta_\epsilon$,

$$S(h; \dot{\mathcal{P}}) = \left\{ \sum_{\substack{i=1 \\ t_i \notin E_\epsilon}}^n + \sum_{\substack{i=1 \\ t_i \in E_\epsilon}}^n \right\} h(t_i)(x_i - x_{i-1})$$

each $t_i \in$ at most two subintervals

$\therefore$ at most $2N_\epsilon$ terms

$$< \sum_{i=1}^n \frac{\epsilon}{2}(x_i - x_{i-1}) + 2N_\epsilon \cdot \delta_\epsilon$$

$$= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

note $S(h; \dot{\mathcal{P}}) \geq 0$, then $|S(h; \dot{\mathcal{P}}) - 0| < \epsilon$.

$\therefore h \in \mathcal{R}[0,1]$ with $\int_0^1 h = 0$. #