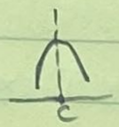
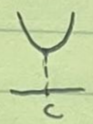
Topic # 2 Mean Value Theorem

Recall: $f: I \to \mathbb{R}$ has



* a relative max at $c \in I$ if $\exists\, \delta > 0$ s.t.

$$f(x) \le f(c), \quad \forall x \in I \cap (c-\delta, c+\delta)$$



* a relative min at $c \in I$ if $\exists\, \delta > 0$ s.t.

$$f(x) \ge f(c), \quad \forall x \in I \cap (c-\delta, c+\delta).$$

* a relative extremum at $c \in I$ if

f has either a relative max or a relative min.

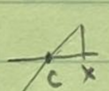
Thm (Interior Extremum Theorem)
If $f: I \to \mathbb{R}$ is differentiable at an interior point c of I at which f has a relative extremum, then

$$f'(c) = 0.$$

(contradiction)

Pf. Assume f has a relative maximum at c and $f'(c) > 0$. Since

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0,$$

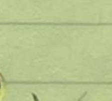


$$\exists\, \delta > 0 \text{ s.t. } \frac{f(x) - f(c)}{x - c} > 0, \quad \forall x \in \big((c-\delta, c+\delta) \setminus \{c\}\big) \cap I$$

Then, for $x \in (c, c+\delta) \cap I$,

$$f(x) - f(c) = \underbrace{\frac{f(x) - f(c)}{x - c}}_{>0} \cdot \underbrace{(x - c)}_{>0} > 0$$

which is a contradiction with the fact that f has a relative max at c.



$\therefore$ It's not true to have $f'(c) > 0$.

Similarly, it's NOT true to have $f'(c) < 0$. (consider $x < c$)

therefore, $f'(c) = 0$. #

Corollary Let $f: I \to \mathbb{R}$ be continuous and have a relative extremum at an interior point c of I, then either $f'(c)$ doest NOT exist or $f'(c) = 0$.

example: ① $f(x) = x^2$, $I = \mathbb{R}$, $c = 0$; $f'(0) = 0$

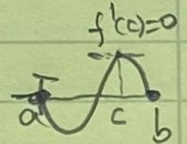
② $f(x) = |x|$, $I = \mathbb{R}$, $c = 0$; $f'(0)$ does NOT exist

In both examples, f has a relative minimum at 0,

in ①: $f'(0) = 0$

in ②: $f'(0)$ does NOT exist.

Thm (Rolle's Theorem)



Let f be continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b) = 0$, then $\exists\, c \in (a, b)$ s.t.

$$f'(c) = 0.$$

Pf. If $f \equiv 0$ on $[a, b]$, then it is obvious to have the conclusion.

Assume f is not identical to zero on (a, b).

Further assume f has some positive values on (a, b); otherwise we consider $-f$ in replace of f.

Since f is continuous on $[a, b]$, f attains its positive value $\sup_{a \le x \le b} f(x)$ at some $c \in [a, b]$.

Since $f(a) = f(b) = 0$, c has to be an exterior point, i.e. $c \in (a, b)$.

Since f is differentiable on (a, b), in particular, at $c \in (a, b)$ it holds by Interior Extremum Theorem that

$$f'(c) = 0.$$

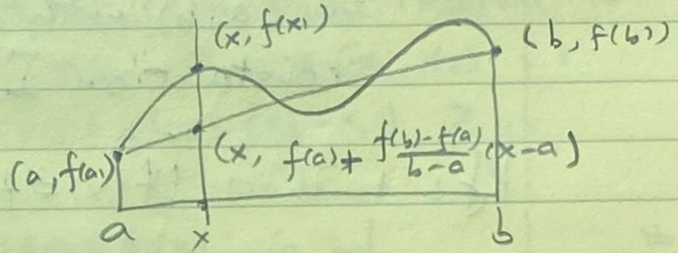
#

Thm (Mean Vaule Theorem)

Let f be continuous on $[a,b]$ and differentiable on (a,b), then $\exists\, c \in (a,b)$ s.t.

$$f(b) - f(a) = f'(c)\,(b-a).$$

Pf: Define



$$\varphi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a),$$

$$a \le x \le b$$

- φ continuous on $[a,b]$
- φ differentiable on (a,b)
- $\varphi(a) = 0 = \varphi(b)$

By Rolle's Theorem, $\exists\, c \in (a,b)$ s.t. $\varphi'(c) = 0$,

i.e. $$0 = \varphi'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

namely, $$f(b) - f(a) = f'(c)\,(b-a).$$ #

RK. Mean Value Theorem tells:

$$\frac{f(b)-f(a)}{b-a} = f'(c);$$

i.e. there is a point on the curve $(x, f(x))$ at which the tangent line is parallel to the line segment through two end points $(a, f(a))$ and $(b, f(b))$. #

Application #1:

$$Df = 0 \Rightarrow f \equiv \text{const}$$

Thm Let f be continuous on $[a,b]$ and differentiable on (a,b) with $f'(x) = 0$, $\forall x \in (a,b)$, then f is constant on I.

Pf: to show: $f(x) = f(a)$, $\forall x \in (a, b]$, Let $a < x \leq b$ be given. Apply Mean Value Theorem to f on $[a, x]$. Then, $\exists c \in (a, x)$ s.t.

$$f(x) - f(a) = f'(c)\,(x-a).$$

By assumption, $f'(c) = 0$, then

$$f(x) - f(a) = 0, \quad \text{i.e.} \quad f(x) = f(a). \quad \#$$

Coro. Let f, g be continuous on $[a,b]$ and differentiable on (a,b) such that

$$f'(x) = g'(x), \quad \forall x \in (a,b),$$

then $\exists$ a constant C such that

$$f = g + C \quad \text{on } [a,b].$$

Pf: apply Thm to $f - g$.

Application #2

$$\text{Monotonicity} \nearrow \searrow \Leftrightarrow Df \gtrless 0$$

Recall: $f: I \to \mathbb{R}$ is

* increasing on I if $f(x_1) \leq f(x_2)$ for any $x_1, x_2 \in I$ with $x_1 < x_2$,
* decreasing on I if $f(x_1) \geq f(x_2)$ for any $x_1, x_2 \in I$ with $x_1 < x_2$
  (or $-f$ is increasing on I)

Thm Let $f: I \to \mathbb{R}$ be differentiable on I, then

(a) f is increasing on I iff $f'(x) \geq 0$, $\forall x \in I$;

(b) f is decreasing on I iff $f'(x) \leq 0$, $\forall x \in I$.

Pf. (a) ($\Leftarrow$) Assume $f'(x) \geq 0$, $\forall x \in I$. Let $x_1, x_2 \in I$ with $x_1 < x_2$. Apply Mean Value Theorem to f on $[x_1, x_2]$, then $\exists\, c \in (x_1, x_2)$ s.t.

$$f(x_2) - f(x_1) = \underbrace{f'(c)}_{\geq 0}\, \underbrace{(x_2 - x_1)}_{>0} \geq 0$$

$$\therefore f(x_1) \leq f(x_2).$$

#

($\Rightarrow$) Assume f is increasing on I. Let $c \in I$,

(f is differentiable at c)

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

note: $\forall x \in I$ with $x \neq c$,

$$\frac{f(x) - f(c)}{x - c} \geq 0.$$

then $f'(c) \geq 0$. #

(b) apply (a) to $-f$. #

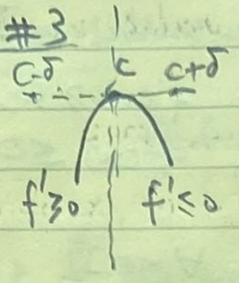
<u>Warning</u>: Sign of f' at one point may NOT be able to imply the monotonicity of f in a neighborhood of that point:

e.g.
$$f(x) = \begin{cases} x + 2x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

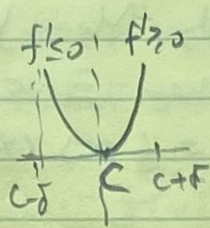
$f'(0) = 1 > 0$, but f is not increasing in any neighborhood of 0.

(Why? $f'(x)$ is NOT continuous at 0!!)

Application #3



f(c): relative max



f(c): relative min

Thm (First Derivative Test for Extrema)
Let f be continuous on $[a,b]$ and differentiable on $(a,c)\cup(c,b)$ with c being an interior point of $[a,b]$.

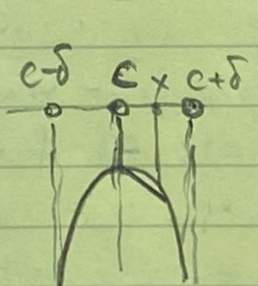
(a) If $\exists\,\delta>0$ s.t. $\begin{cases} f'(x)\geq 0, & \forall x\in(c-\delta,c) \\ f'(x)\leq 0, & \forall x\in(c,c+\delta), \end{cases}$

then f has a relative maximum at c.

(b) If $\exists\,\delta>0$ s.t. $\begin{cases} f'(x)\leq 0, & \forall x\in(c-\delta,c) \\ f'(x)\geq 0, & \forall x\in(c,c+\delta) \end{cases}$

then f has a relative minimum at c.

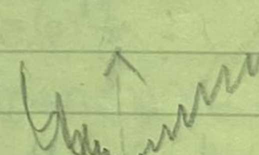
Pf. (a)



$$f(x)-f(c)=\underbrace{f'(c_x)}_{\leq 0}\underbrace{(x-c)}_{>0}\leq 0$$

#

RK: Converse may not be true



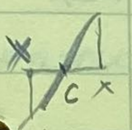
Exercise 9

Application # 4 Intermediate Value Property of Derivative

i.e. $f' \in (A, B) \Rightarrow f'$ can take all values between A and B

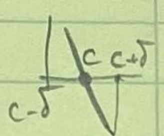
note: no need to assume f' is continuous.

Lemma Let $f: I \to \mathbb{R}$ be differentiable at $c \in I$



(a) If $f'(c) > 0$ then $\exists\, \delta > 0$ s.t.

$$f(x) > f(c), \quad \forall x \in (c, c+\delta) \cap I;$$
$$f(x) < f(c), \quad \forall x \in (c-\delta, c) \cap I.$$



(b) If $f'(c) < 0$ then $\exists\, \delta > 0$, s.t.

$$f(x) < f(c), \quad \forall x \in (c, c+\delta) \cap I;$$
$$f(x) > f(c), \quad \forall x \in (c-\delta, c) \cap I.$$

Pf: (a) $f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c} > 0$

then $\exists\, \delta > 0$, s.t. if $0 < |x - c| < \delta$, then

$$\frac{f(x) - f(c)}{x - c} > \frac{1}{2} f'(c) > 0$$

$$\therefore f(x) - f(c) = \underbrace{\frac{f(x) - f(c)}{x - c}}_{>0} (x - c) \begin{cases} > 0, & 0 < x - c < \delta \\ < 0 & -\delta < x - c < 0 \end{cases}$$

#

(b) apply (a) to $-f$. #

Thm (Darboux's Theorem)

Let $f: I \to \mathbb{R}$ be differentiable on I and k be a number between $f'(a)$ and $f'(b)$, then $\exists\, c \in (a, b)$ s.t. $f'(c) = k$.

Pf: WLG, assume $f'(a) < k < f'(a)$. Define

$$g(x) = kx - f(x), \quad a \le x \le b.$$

then g is differentiable on $[a,b]$. In particular, g is continuous on $[a,b]$, then g attains its maximum on $[a,b]$.

<u>Claim:</u> g can not attain the maximum at a or b.

if so, $\exists\, c \in (a,b)$ at which g attains the max
Since g is differentiable at c, Interior Extremum Theorem implies that $g'(c) = 0$, i.e

$$0 = g'(c) = k - f'(c).$$

so, $f'(c) = k$. #

<u>Pf of claim:</u>
g is differentiable at a,
so $g'(a) = k - f'(a) < 0$,
then g can not attain its max at a;
Similarly, g is diff'ble at b
so $g'(b) = k - f'(b) > 0$
then g also can NOT attain its max at b.

#

<u>RK</u>: Thm tells that if a function g on $[a,b]$ does not satisfy the intermediate value property, then g can NOT be the derivative on $[a,b]$ of any function.

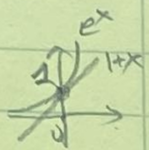
<u>eg</u>: $g(x) = \operatorname{sign}(x) = \begin{cases} +1 & 0 < x \le 1 \\ 0 & x = 0 \\ -1 & -1 \le x \le 0 \end{cases}$

NO intermediate value property.
($g(1) = 1$, $g(-1) = -1$, no $x \in (-1,1)$ s.t. $g(x) = k$
as long as $-1 \le k \le 1$ with $k \ne 0$).

then no function f (diff'able on $[-1,1]$) s.t.

$$f'(x) = g(x), \quad a \le x \le b. \qquad \#$$

Application # Prove Inequalities.



(1) Show: $e^x \geq 1+x$, $x \in \mathbb{R}$, with "=" iff $x=0$.

Pf: if $x=0$, "=" holds

if $x>0$, $\quad e^x - 1 = e^x - e^0$

$= e^{c_x}(x-0) \qquad (\exists\, c_x \in (0,x))$

$> x$

if <0 $\quad e^x - 1 = e^x - e^0$

$= e^{c_x}(x-0) \qquad (\exists\, c_x \in (x,0))$

$> x \qquad 0 < e^{c_x} < 1$

$x e^{c_x} > x$

#

(2) Show: $\sin x \le x$, $\forall x \geq 0$

Pf $\quad \sin x = \sin x - \sin 0$

$= (\cos c_x)(x-0) \qquad (\exists\, c_x \in [0,x])$

$\le x. \qquad -1 \le \cos c_x \le 1$

$-x \le x \cos c_x \le x$

#

(3) Let $\alpha > 1$, show: $(1+x)^\alpha \geq 1 + \alpha x$, $\forall x > -1$, with "=" iff $x=0$

Pf $\quad (1+x)^\alpha - 1 = \alpha (1+c_x)^{\alpha-1} x$

$f(x) = (1+x)^\alpha$

$f'(x) = \alpha(1+x)^{\alpha-1}$

c_x is between 0 and x

Case $x>0$: $\quad 0 < c_x < x$

$(1+c_x)^{\alpha-1} > 1$, then $(1+x)^\alpha - 1 > \alpha x$

Case $-1<x<0$: $-1<x<c_x<0$

$0 < (1+c_x)^{\alpha-1} < 1$

$\alpha x (1+c_x)^{\alpha-1} > \alpha x$, then $(1+x)^\alpha - 1 > \alpha x$

Case $x=0$: "=" holds.

#