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Topic #4 Taylor's Theorem

Def:

$$P_n(x) \overset{\text{def.}}{=} f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

n^{th} Taylor polynomial for f at x_0

Notation:

① $f^{(k)}(x_0)$ is the k^{th} derivative of f at x_0
(Remark: this requires that the $(k-1)^{th}$ derivative $f^{(k-1)}(x)$ exists in an interval containing x_0)

② Convention: $f^{(0)}(x_0) = f(x_0)$, $0! = 1$

Fact: $P_n^{(k)}(x_0) = f^{(k)}(x_0)$, $k = 0, 1, \cdots, n$
i.e. P_n and its derivatives up to order n agree with f and its derivatives up to order n at x_0, respectively.

Taylor's Theorem:
Assume that $f: I=[a,b] \to \mathbb{R}$ and it derivatives up to order n $f', f'', \cdots, f^{(n)}$ $(n \in \mathbb{N})$ are continuous on I and $f^{(n+1)}$ exists on (a,b). Let $x_0, x \in I$, then $\exists$ a point c $(=c_{x_0,x})$ between x_0 and x such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

is the n^{th} Taylor polynomial for f at x_0.

Proof: Take $x_0, x \in I = [a,b]$.

WLG, assume $x \neq x_0$, otherwise it is trivial.

Let J be the closed interval with end points x_0 and x.

Define $F: J \to \mathbb{R}$ by

$$F(t) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k,$$

$$= f(x) - \left[f(t) + f'(t)(x-t) + \cdots + \frac{f^{(n)}(t)}{n!}(x-t)^n\right], \quad t \in J.$$

We see that F is differentiable on J and

$$F'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n, \quad t \in J.$$

Further define $G: J \to \mathbb{R}$ by

$$G(t) = F(t) - \left(\frac{x-t}{x-x_0}\right)^{n+1} F(x_0), \quad t \in J.$$

Verify:

$$G(x_0) = F(x_0) - \left(\frac{x-x_0}{x-x_0}\right)^{n+1} F(x_0) = F(x_0) - F(x_0) = 0$$

$$G(x) = F(x) = f(x) - f(x) = 0$$

Apply Rolle's Theorem, $\exists\, c \in J$ (between x_0 and x) s.t.

$$0 = G'(c) = F'(c) + (n+1)\frac{(x-c)^n}{(x-x_0)^{n+1}} F(x_0)$$

this implies

$$f(x) - P_n(x) = F(x_0)$$

$$= -\frac{1}{n+1} \frac{(x-x_0)^{n+1}}{(x-c)^n} F'(c)$$

$$= +\frac{1}{n+1} \frac{(x-x_0)^{n+1}}{\cancel{(x-c)^n}} \cdot \frac{\cancel{(x-c)^n}}{n!} f^{(n+1)}(c) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

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RK: $f(x) = P_n(x) + R_n(x)$

with $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$ remainder

Lagrange form (or derivative form)

Future (Chapter 7): integral form of remainder

Applications

A#1 Find approximate values with given errors.

e.g. 1. $f(x) = (1+x)^{\frac{1}{3}}$, $x > -1$, $x_0 = 0$

2^{nd} order Taylor polynomial for $f(x)$ at $x_0 = 0$:

$$P_2(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2$$

$$f(x) = (1+x)^{\frac{1}{3}}, \quad f(0) = 1$$
$$f'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}}, \quad f'(0) = \frac{1}{3}$$
$$f''(x) = -\frac{2}{9}(1+x)^{-\frac{5}{3}}, \quad f''(0) = -\frac{2}{9}$$

$$\therefore P_2(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2$$

For any $x > -1$, $\exists$ c between 0 and x such that

$$f(x) = P_2(x) + R_2(x)$$

Where $R_2(x) = \frac{f'''(c)}{3!}(x-x_0)^3$

$$= \frac{1}{3!}\underbrace{\left[\left(-\frac{2}{9}\right)\left(-\frac{5}{3}\right)(1+c)^{-\frac{8}{3}}\right]}_{f^{(3)}(c)}x^3$$

$$= \frac{5}{81}(1+c)^{-\frac{8}{3}}x^3.$$

For instance, $x = 0.3$, then

$$P_2(0.3) = 1 + \frac{1}{3}\cdot 0.3 - \frac{1}{9}\cdot(0.3)^2 = 1.09$$

$$R_2(0.3) = \frac{5}{81}(1+c)^{-\frac{8}{3}}(0.3)^3$$

$$\Rightarrow 0 < R_2(0.3) \le \frac{5}{81}\cdot(0.3)^3 \leftarrow \left[\begin{array}{l} \because 0<c<0.3 \\ \because 0<(1+c)^{-\frac{8}{3}}<1 \end{array}\right]$$

$$= \frac{1}{600} < 0.0017$$

$$\Rightarrow \quad 0 < f(0.3) - P_2(0.3) = R_2(0.3) < 0.0017$$

$$\text{i.e.} \quad 0 < \sqrt[3]{1.3} - 1.09 < 0.0017$$

$\therefore$ $\sqrt[3]{1.3} \sim 1.09$ up to two decimal places.

e.g. 2. Find approximate value of e with error $< 10^{-5}$ (5 decimal places)

"given error"

Let $f(x) = e^x$, $x_0 = 0$.

$f'(x) = e^x$; $f^{(k)}(x) = e^x$, $k = 1, 2, \ldots$

$f(x) = P_n(x) + R_n(x)$

with $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$

$$= \frac{e^c}{(n+1)!}x^{n+1} \quad \text{for } c \text{ between } 0 \text{ and } x.$$

Take $x = 1$, then $e = P_n(1) + \dfrac{e^c}{(n+1)!}$, $0 < c < 1$.

Where $0 < R_n(1) \le \dfrac{e}{(n+1)!} < \dfrac{3}{(n+1)!}$

To ensure error $< 10^{-5}$, we need

$$\frac{3}{(n+1)!} < 10^{-5}, \quad \text{i.e. } (n+1)! > 3\cdot 10^5 = 300{,}000$$

note: $8! = 40{,}320$, $9! = 362{,}320$

then $\min\{n \geq 1 : (n+1)! > 3 \cdot 10^5\} = 8$.

hence,

$$e = P_8(1) + R_8(1)$$
$$= f(0) + \frac{f'(0)}{1!}1^1 + \dots + \frac{f^{(8)}(0)}{8!}1^8 + R_8(1)$$
$$= \left(1 + 1 + \frac{1}{2!} + \dots + \frac{1}{8!}\right) + R_8(1)$$
$$= 2.\underbrace{71828}78\dots + R_8(1)$$

with $0 < R_8(1) < 10^{-5}$

$\therefore$ $e \approx 2.71828$ up to 5 decimals #

A#2 to prove inequalities

e.g. 1 Show: $1 - \frac{1}{2}x^2 \leq \cos x$, $\forall x \in \mathbb{R}$

Pf. Let $f(x) = \cos x$, $x_0 = 0$.

Taylor's Thm $\Rightarrow$ $\cos x = 1 - \frac{1}{2}x^2 + R_2(x)$

with $R_2(x) = \frac{f^{(3)}(c)}{3!}x^3 = \frac{\sin c}{6}x^3$

for some c between 0 and x

Case: $x = 0$. "=" holds.

Case: $0 < x < \pi$. Then, $0 < c < x < \pi$, hence $\sin c > 0$, $x^3 > 0$

then $R_2(x) > 0$

$\therefore$ $1 - \frac{1}{2}x^2 < \cos x$.

Case: $-\pi < x < 0$. Then, $-\pi < x < c < 0$,

hence, $\sin c < 0$, $x^3 < 0$, $\therefore$ $R_2(x) > 0$

$\therefore$ $1 - \frac{1}{2}x^2 < \cos x$.

Case $|x| \geq \pi$. Then

$$1 - \frac{1}{2}x^2 \leq 1 - \frac{1}{2}\pi^2 < -1 \leq \cos x$$

i.e. $1 - \frac{1}{2}x^2 < \cos x$.

RK. "=" holds iff $x = 0$.

e.g. 2. Show: $\forall k = 1, 2, \dots, \ \forall x > 0,$

$$x - \frac{1}{2}x^2 + \dots - \frac{1}{2k}x^{2k} < \ln(1+x) < x - \frac{1}{2}x^2 + \dots + \frac{1}{2k+1}x^{2k+1}$$

Pf: Let $f(x) = \ln(1+x)$ with $x > -1$, $x_0 = 0$.
Compute

$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = -\frac{1}{(1+x)^2}$$

$$\dots$$

$$f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}, \quad n = 1, 2, \dots$$

hence $f^{(n)}(0) = (-1)^{n-1}(n-1)!$

$\therefore$ n^{th} Taylor polynomial of $f(x)$ at $x_0 = 0$ is

$$P_n(x) = 0 + 1\cdot x - \frac{1}{2!}\cdot x^2 + \frac{1}{3!}\cdot 2!\, x^3 - \dots + \frac{1}{n!}(-1)^{n-1}(n-1)!\, x^n$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots + \frac{(-1)^{n-1}}{n}x^n$$

$$n = 1, 2, \dots$$

Then $f(x) = P_n(x) + R_n(x)$ with the remainder

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^{n+1}$$

$$= \frac{(-1)^n n!}{(1+c)^{n+1}} \cdot \frac{x^{n+1}}{(n+1)!}$$

for some c between 0 and x.

Let $x > 0$, then $0 < c < x$, hence

$$R_n(x) = \frac{(-1)^n}{n+1}\left(\frac{x}{1+c}\right)^{n+1} \begin{cases} > 0 & \text{if } n \text{ even} \\ < 0 & \text{if } n \text{ odd} \end{cases}$$

In particular,
for $n = 2k$, $\ln(1+x) = P_{2k}(x) + R_{2k}(x) > P_{2k}(x)$

i.e. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots - \frac{x^{2k}}{2k}$ $(\forall x > 0)$

For $n=2k+1$, $\ln(1+x) = P_{2k+1}(x) + R_{2k+1}(x) < P_{2k+1}(x)$

i.e. $\ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{x^{2k+1}}{2k+1}$ $(\forall x>0)$.

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e.g. 3 Show: $e^{\pi} > \pi^{e}$.

Pf: Taylor's Thm $\Rightarrow$ $e^x = 1 + x + R_1(x)$

with $R_1(x) = \frac{e^c}{2!}x^2 > 0$

for some c between 0 and x $\neq 0$

[note: $e^c > 0$, $\forall c \in \mathbb{R}$]

$\therefore$ $e^x > 1 + x$, $\forall x \neq 0$

plug $x = \frac{\pi}{e} - 1 > 0$ (We know approxation values of π and e

$\pi \approx 3.14\cdots$

$e \approx 2.7\cdots$

then $e^{\frac{\pi}{e}-1} > 1 + \left(\frac{\pi}{e} - 1\right) = \frac{\pi}{e}$

i.e. $e^{\frac{\pi}{e}} > \pi$

i.e. $e^{\pi} > \pi^{e}$. #

A#3 Higher order derivative test for relative extrema

Thm Let $f: I \to \mathbb{R}$ and its derivatives up to order n f', f'', $\ldots$ $f^{(n)}$ be continuous in a neigh'd of an interior point x_0 of I (interval) and

"essential"

$$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0 \quad \text{but } f^{(n)}(x_0) \neq 0,$$

then

(1) if n even & $f^{(n)}(x_0) > 0$, then f has a relative min at x_0;

(2) if n even & $f^{(n)}(x_0) < 0$, then f has a relative max at x_0;

(3) if n odd then f has neither a relative max nor a relative min at x_0.

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RK: For $n=2$, thm is 2^{nd} Derivative Test.

Pf. Taylor's Thm gives:

$$f(x) = f(x_0) + \cdots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1} + \frac{f^{(n)}(c)}{n!}(x-x_0)^n$$

for some c between x_0 and x.

Apply assumptions $f'(x_0) = \cdots = f^{(n-1)}(x_0) = 0$,

$$\therefore f(x) = f(x_0) + \frac{f^{(n)}(c)}{n!}(x-x_0)^n$$

Case (1): n even, $f^{(n)}(x_0) > 0$.

$\because$ $f^{(n)}(x)$ is continuous at x_0; x_0 is an interior pt of I

$\therefore$ $\exists \delta > 0$ s.t. $f^{(n)}(x) > 0$, $\forall x \in (x_0-\delta, x_0+\delta) \subset I$.

then for $x \in (x_0-\delta, x_0+\delta)$, c is between x_0 and x,

in particular, $c \in (x_0-\delta, x_0+\delta)$, then

$$f^{(n)}(c) > 0.$$

then

$$f(x) = f(x_0) + \frac{f^{(n)}(c)}{n!}(x-x_0)^n \geq f(x_0),$$

$x \in (x_0-\delta, x_0+\delta)$.

(note: $(x-x_0)^n \geq 0$ for n even)

i.e. f has a relative min at x_0.

Case (2): n even, $f^{(n)}(x_0) < 0$.

Follow the similar process,

$$f(x) = f(x_0) + \frac{f^{(n)}(c)}{n!}(x-x_0)^n \leq f(x_0)$$

$x \in (x_0-\delta, x_0+\delta)$

(still: $(x-x_0)^n \geq 0$ for n even)

similar

Case (3): n odd,

$\exists \delta > 0$ s.t. $\text{sgn}(f^{(n)}(x)) = \text{sgn}(f^{(n)}(x_0)) \neq 0$,

$\forall x \in (x_0-\delta, x_0+\delta) \subset I$,

then

$$f(x) - f(x_0) = \frac{f^{(n)}(c)}{n!}(x-x_0)^n$$

has different signs on $(x_0-\delta, x_0)$ and $(x_0, x_0+\delta)$.

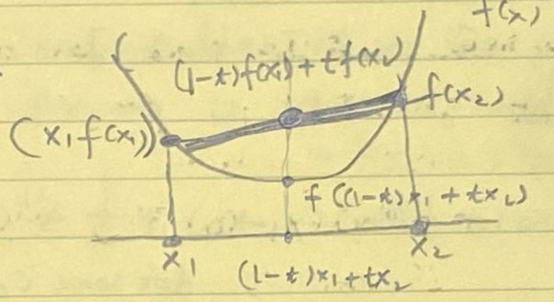
$\therefore$ neither relative max, nor relative min. #

A#4: Convexity $\overset{f'' \text{ exist in } (a,b)}{\Longleftrightarrow}$ $f'' \geq 0$.

Def. $f: I \to \mathbb{R}$ is convex on an interval I if

$$f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + t f(x_2),$$
$$\forall t \in [0,1], \quad \forall x_1, x_2 \in I.$$

Note:



chord joining any two points $(x_1, f(x_1))$ $(x_2, f(x_2))$ lies above the graph of f.

e.g. $f(x) = |x|$ is convex on $\mathbb{R}$.
note: f is NOT differentiable at 0.

Thm: Let $f: I \to \mathbb{R}$ for an open interval I have the second derivative f'' on I, then

f is convex on I iff $f''(x) \geq 0$, $\forall x \in I$.

Pf: "only if" $\Rightarrow$: Assume f is convex on I. Let $x \in I$ (open interval).

$\because$ $f''(x)$ exists

$$\therefore f''(x) = \lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \quad \text{(Exercise)}$$

Note: "f is convex on I" gives:

$\forall h \in \mathbb{R}$ with $x \pm h \in I$,

$$f(x) = f\left(\tfrac{1}{2}(x+h) + \tfrac{1}{2}(x-h)\right) \leq \tfrac{1}{2} f(x+h) + \tfrac{1}{2} f(x-h)$$

i.e. $2f(x) \leq f(x+h) + f(x-h)$

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thus $\dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2} \geq 0$,

$\forall h \in \mathbb{R}$ s.t. $\begin{cases} h \neq 0 \\ x \pm h \in I \end{cases}$

$\therefore f''(x) = \lim_{h\to 0} \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2} \geq 0.$

"$\Leftarrow$" Assume: $f''(x) \geq 0$, $\forall x \in I$.

Let $0 \leq t \leq 1$, $x_1, x_2 \in I$.

Define $x_0 = (1-t)x_1 + tx_2$.

Obvious to see $x_0 \in I$ since I is an open interval

Taylor's Thm gives: $f(x) = P_n(x) + R_n(x)$ — n^{th} Taylor polynomial

At x_1: $f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + \boxed{\frac{1}{2} f''(c_1)(x_1 - x_0)^2}$

for some c_1 between x_0, x_1 (still in I)

$\geq f(x_0) + f'(x_0)(x_1 - x_0)$

and

At x_2: $f(x_2) = f(x_0) + f'(x_0)(x_2 - x_0) + \boxed{\frac{1}{2} f''(c_2)(x_2 - x_0)^2}$

for some c_2 between x_0, x_2 (still in I)

$\geq f(x_0) + f'(x_0)(x_2 - x_0)$

Therefore, we

$(1-t)f(x_1) + tf(x_2)$

$\geq (1-t)f(x_0) + tf(x_0)$

$+ f'(x_0)\left[(1-t)(x_1 - x_0) + t(x_2 - x_0)\right]$

$= f(x_0) + f'(x_0)\left[\cancel{(1-t)x_1 + tx_2 - x_0}\right]$ gone

($\because x_0 = (1-t)x_1 + tx_2$.

$= f(x_0)$. #

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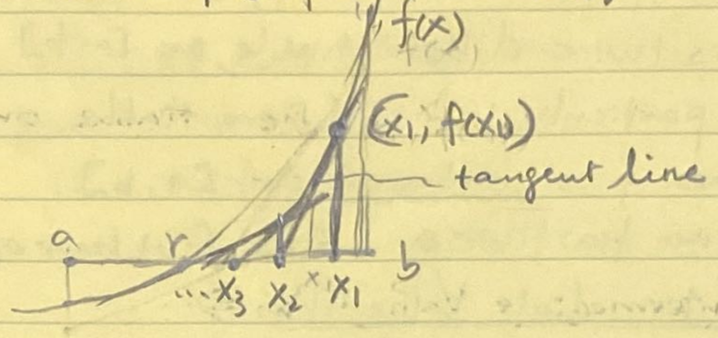
A#5. Newton's Method

Find the zeros of a function: $f(r)=0$

idea:



$f(a)f(b)<0$

Tangent line at x_1: $y-f(x_1)=f'(x_1)(x-x_1)$

Assume it intersect x-axis at x_2, then

$$-f(x_1)=f'(x_1)(x_2-x_1)$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

provided $f'(x_1)\neq 0$.

Repeat the process: $\forall n\geq 1$,

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \quad \text{provided } f'(x_k)\neq 0, \ k=1,\cdots,n$$

Uder certain conditions:

$$x_n \xrightarrow[n\to\infty]{} r \quad \text{such that } f(r)=0.$$

Thm (Newton's method)

Let
- $f:[a,b]\to\mathbb{R}$ twice differentiable $(a<b)$
- $f(a)f(b)<0$ (i.e. $f(a)$, $f(b)$ have opposite signs)
- $\exists$ constants $m>0$, $M\geq 0$ such that

$$|f'(x)|\geq m>0, \quad \forall x\in[a,b]$$
$$|f''(x)|\leq M, \quad \forall x\in[a,b].$$

Then, $\exists$ a subinterval I^* containing a zero r of f s.t. $\forall x_1\in I^*$, the sequence (x_n) defined by

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}, \quad n=1,2,\cdots$$

is in I^* with $\lim_{n\to\infty} x_n = r$. Moreover,

$$|x_{n+1} - r| \le K\, |x_n - r|^2, \quad n = 1, 2, \cdots$$

where $K = M/2m$

step 1. Find r.

Pf. f is twice differentiable on $[a, b]$

in particular, f is differentiable on $[a, b]$

then f is continuous on $[a, b]$

Since $f(a)f(b) < 0$, $f(a), f(b)$ have opposite signs,

Intermediate Value Thm $\Rightarrow$

$$\exists\, r \in (a, b), \text{ s.t. } f(r) = 0$$

Note: $|f'(x)| \ge m > 0,\ \forall x \in [a, b]$

then by Rolle's Thm,

r is the unique zero of f on $[a, b]$

i.e. $f(x) \ne 0,\ \forall x \in [a, b] \setminus \{r\}$

step 2. Find I^* (Exercise: By contradiction)

Let $x' \in I$, Taylor's Thm gives

$$0 = f(r) = f(x') + f'(x')(r - x') + \frac{f''(c')}{2}(r - x')^2$$

for some c' between r and x'.

Define $x'' = x' - \dfrac{f(x')}{f'(x')}$, then

$$x'' = x' + \frac{f'(x')(r - x') + \frac{f''(c')}{2}(r - x')^2}{f'(x')}$$

$$= r + \frac{1}{2} \frac{f''(c')}{f'(x')}(r - x')^2$$

imply

$$|x'' - r| \le \frac{1}{2} \frac{|f''(c')|}{|f'(x')|}(r - x')^2$$

$$\le \frac{1}{2} \frac{M}{m} |x' - r|^2 = K\, |x' - r|^2 \quad (*)$$

Choose $\delta > 0$ s.t.

$\delta < \frac{1}{K}$ (i.e. $K\delta < 1$) and $[r - \delta, r + \delta] \subset [a, b]$

(r is an interior p't)

and let $I^* = [r-\delta, r+\delta]$.

Step 3. Let $x_1 \in I^*$, verify (x_n) in I^*.

(Induction on n:) Let $x_n \in I^*$ for some $n \geq 1$, then

then $(*)$ gives:

$$
\begin{aligned}
|x_{n+1} - r| &\leq k\,|x_n - r|^2 \\
&\leq k\delta\,|x_n - r| \\
&\leq |x_n - r| \\
&\leq \delta,
\end{aligned}
$$

$0<k\delta<1$

i.e. $x_{n+1} \in I^*$.

then by induction, $(x_n \in I^*,\ n = 1, 2, \cdots$

therefore by $(*)$, (x_n) in I^* satisfies

$$|x_{n+1} - r| \leq k\,|x_n - r|^2, \quad n = 1, 2, \cdots$$

Step 4: Convergence of (x_n):

We do iteration: $\forall n \geq 1$,

$$
\begin{aligned}
|x_{n+1} - r| &\leq k\delta\,|x_n - r| \\
&\leq (k\delta)^2\,|x_{n-1} - r| \\
&\leq \cdots \\
&\leq (k\delta)^n\,|x_1 - r|
\end{aligned}
$$

$\because 0 < k\delta < 1$

$\therefore (k\delta)^n \to 0$ as $n \to \infty$,

since $|x_1 - r|$ is finite,

$$|x_{n+1} - r| \to 0 \text{ as } n \to \infty$$

i.e. $\lim_{n\to\infty} x_n = r$. #

Example: Approximate $\sqrt{2}$ by Newton's Method.

Sol: $\sqrt{2}$ is the zero of $f(x) = x^2 - 2$, $x \in \mathbb{R}$.

$f'(x) = 2x$, $f'(x) = 0$ only if $x = 0$.

note: $f(1) = -1 < 0$, $f(2) = 2 > 0$.

take: $I = [1, 2]$, guess: $x_1 = 1$,

Note: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{x_n^2 - 2}{2x_n}$$

$$= x_n - \tfrac{1}{2}x_n + \frac{1}{x_n}$$

$$= \tfrac{1}{2}x_n + \frac{1}{x_n}$$

$x_1 = 1 \Rightarrow x_2 = \frac{1}{2}\cdot 1 + \frac{1}{1} = \frac{3}{2} = 1.5$

$x_3 = \frac{1}{2}\cdot\frac{3}{2} + \frac{2}{3} = \frac{17}{12} \approx 1.416\cdots$

$\vdots$

$x_5 \approx 1.41421356237\,4$

(correct to 11 decimals)

RK ① Rewrite $|x_{n+1} - r| \le K\,|x_n - r|^2$ as

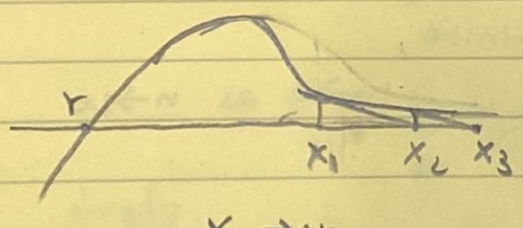
$$K\,|x_{n+1} - r| \le (K\,|x_n - r|)^2.$$

i.e. if $K\,|x_n - r| < \epsilon$ (small)

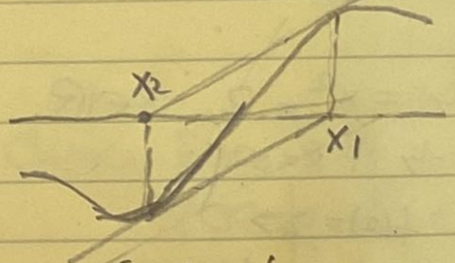
then $K\,|x_{n+1} - r| \le \epsilon^2$ (smaller "quadratically")

the sequence (x_n) generated by Newton's Method is said to converge "quadratically"

RK ② Be "smart" to choose x_1:



$x_n \to \infty$



$(x_n) = (x_1, x_2, x_1, \cdots)$ no limit!