MATH2060AB Homework 8
Reference Solutions

8.1.21. Let $\epsilon > 0$. Since $(f_n), (g_n)$ converge uniformly on A to f, g, respectively, there exist $K_1, K_2 > 0$ such that the following holds. If $n > K_1$ and $x \in A$, then $|f_n(x) - f(x)| < \epsilon/2$. If $n > K_2$ and $x \in A$, then $|g_n(x) - g(x)| < \epsilon/2$. Let $K = \max\{K_1, K_2\}$, then for $n > K$ and $x \in A$, it holds that

$$|(f_n + g_n)(x) - (f + g)(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \epsilon.$$

Hence, $(f_n + g_n)$ converges uniformly on A to $f + g$.

8.1.22. Let $\epsilon > 0$. Choose $N := [1/\epsilon] + 1$. Then for $n > N$ and $x \in \mathbb{R}$, we have $|f_n(x) - f(x)| = 1/n < \epsilon$. Hence, f_n converges uniformly on $\mathbb{R}$ to f.

A direct calculation shows that $|f_n^2(x) - f^2(x)| = |\frac{2x}{n} + \frac{1}{n^2}|$. Let $\epsilon_0 = 1$, then for any $k \in \mathbb{N}$, let $n_k = k$ and $x_k = k$, then

$$|f_{n_k}^2(x_k) - f^2(x_k)| = |\frac{2x_k}{n_k} + \frac{1}{n_k^2}| = 2 + \frac{1}{k^2} > 1 = \epsilon_0.$$

Thus, (f_n^2) does not converge uniformly on $\mathbb{R}$.

8.2.2. One can see that for each $n \in \mathbb{N}$, f_n is continuous on $[0,1]$ and $f_n \to f \equiv 0$ on $[0,1]$, where the limit function f is also continuous on $[0,1]$. We prove that the convergence is not uniform. In fact, let $\epsilon_0 = 1$. For $k \in \mathbb{N}$, let $n_k = 2k$ and $x_k = \frac{1}{2k} \in [0,1]$, then $0 < x_k \leq \frac{1}{n_k}$ so that

$$|f_{n_k}(x_k) - f(x_k)| = |n_k^2 x_k - 0| = 2k > 1 = \epsilon_0, \quad \forall\, k \in \mathbb{N}.$$

Hence, the convergence is not uniform.

8.2.5. Let $\epsilon > 0$. Since $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$, there exists $\delta > 0$ such that if $x, y \in \mathbb{R}$ and $|x_1 - x_2| < \delta$, then it holds that $|f(x_1) - f(x_2)| < \epsilon$. We let $N = [1/\delta] + 1$. Then for any $n > N$ and $x \in \mathbb{R}$, one has $|(x + 1/n) - x| \leq \delta$, which yields

$$|f_n(x) - f(x)| = |f(x + \frac{1}{n}) - f(x)| < \epsilon.$$

Hence, (f_n) converges uniformly on $\mathbb{R}$ to f.

8.2.7. For each $n \in \mathbb{N}$, the fact that f_n is bounded implies that there exists $M_n > 0$ such that $\sup_{x \in A} |f_n(x)| \leq M_n$. By the uniform convergence, there exists $K \in \mathbb{N}$ such that for $n \geq K$ and $x \in A$, it holds that $|f_n(x) - f(x)| < 1$. It then follows that $|f(x)| \leq |f(x) - f_K(x)| + |f_K(x)| \leq 1 + |f_K(x)| \leq 1 + M_K < \infty$, for all $x \in A$. Hence, f is bounded on A.

8.2.13. Let $f_n(x) = \frac{\sin nx}{nx}$ if $x \neq 0$ and 1 otherwise, then for each $n \in \mathbb{N}$, f_n is continuous on $\mathbb{R}$ and hence $\int_a^\pi (\sin nx)/(nx)dx = \int_a^\pi f_n(x)dx$ exists. It is also direct to see that $f_n \to f$ on $\mathbb{R}$ with $f(x) = 0$ if $x \neq 0$ and 1 otherwise. Thus f is integrable in the interval with ending points a and π and $\int_a^\pi f = 0$. Note that f_n is uniformly bounded on any finite interval $[-A, A]$ with $A > 0$. In fact,

$$\|f_n\|_{[-A,A]} = \sup_{|x| \leq A} |f_n(x)| = \sup_{0<|x|\leq A} |f_n(x)| = \sup_{0<|x|\leq A} |\frac{\sin nx}{nx}| \leq \max\{\sup_{0<|y|\leq A} |\frac{\sin y}{y}|, \frac{1}{A}\} < \infty,$$

for all $n \in \mathbb{N}$. Then, Bounded Convergence Theorem implies that $\lim\limits_{n\to\infty} \int_a^\pi f_n = \int_a^\pi f = 0$ for any $a \in \mathbb{R}$.

8.2.16. (a) For each $n \in \mathbb{N}$, $f_n = 0$ on $[0,1]$ except for a finite number of points $r_1, ..., r_n$ in $[0,1]$. Since $0 \in \mathcal{R}[0,1]$, it holds that $f_n \in \mathcal{R}[0,1]$. (b) It suffices to show $f_n \leq f_{n+1}$ on $[0,1]$ for each $n \in \mathbb{N}$. In fact, if $x \in \{r_1, ..., r_n\}$ then $f_n(x) = 1 = f_{n+1}(x)$; if $x = r_{n+1}$ then $f_n(x) = 0 < 1 = f_{n+1}(x)$; otherwise, for any $x \in [0,1] - \{r_1, ..., r_{n+1}\}$, $f_n(x) = 0 = f_{n+1}(x)$. (c) Since f_n is bounded on $[0,1]$, Monotone Convergence Theorem implies that the sequence $(f_n(x))$ is convergent for each $x \in [0,1]$ and we set $f := \lim f_n$ on $[0,1]$. If $x \in [0,1]$ is irrational then $f_n(x) = 0$ for all $n \in \mathbb{N}$ and hence $f_n(x) \to 0$; if $x \in [0,1]$ is rational then there is $n_0 \in \mathbb{N}$ such that $x = r_{n_0}$ and hence, for $n \geq n_0$, $f_n(x) = f_n(r_{n_0}) = 1$, then $f_n(x) \to 1$. Combining both cases, the limit function f satisfies that $f(x) = 0$ if $x \in [0,1]$ is irrational and $f(x) = 1$ if $x \in [0,1]$ is rational. Therefore, f is the Dirichlet function, which is not Riemann integrable on $[0,1]$.