MATH2060AB Homework 3
Reference Solutions

6.4.3. For convenience, denote

$$C_n^k := \binom{n}{k}.$$

When $n = 0$, the equality obviously holds. By induction process, we assume

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} C_n^k f^{(n-k)}(x) g^{(k)}(x),$$

for $n = m$. Then when $n = m+1$,

$$\begin{aligned}
(fg)^{(m+1)}(x) &= \big((fg)^{(m)}(x)\big)' = \big(\sum_{k=0}^{m} C_m^k f^{(m-k)}(x) g^{(k)}(x)\big)' \\
&= \sum_{k=0}^{m} C_m^k \big(f^{(m+1-k)}(x) g^{(k)}(x) + f^{(m-k)}(x) g^{(k+1)}(x)\big) \\
&= f^{(m+1)}(x) g^{(0)}(x) + f^{(0)}(x) g^{(m+1)}(x) + \sum_{k=1}^{m} (C_m^k + C_m^{k-1}) f^{(m-k+1)}(x) g^{(k)}(x) \\
&= f^{(m+1)}(x) g^{(0)}(x) + f^{(0)}(x) g^{(m+1)}(x) + \sum_{k=1}^{m} C_{m+1}^k f^{(m-k+1)}(x) g^{(k)}(x) \\
&= \sum_{k=0}^{m+1} C_{m+1}^k f^{(m+1-k)}(x) g^{(k)}(x).
\end{aligned}$$

6.4.7. Direct calculations by Taylor expansion show that $(1+x)^{\frac{1}{3}} = P_2(x) + R_2(x)$ where

$$P_2(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2$$

and

$$R_2(x) = \frac{1}{6}\frac{10}{27}(1+\xi)^{-8/3} x^3 < \frac{5}{81}x^3,$$

for some $0 < \xi < x$. We let $x = 0.2$ and $x = 1$ respectively to get $|(1.2)^{\frac{1}{3}} - \frac{239}{225}| < \frac{1}{2025}$ and $|2^{\frac{1}{3}} - \frac{11}{9}| < \frac{5}{81}$.

6.4.8. For fixed x_0, x, we get by Taylor's Theorem that

$$R_n(x) = \frac{1}{(n+1)!} e^c (x - x_0)^{n+1},$$

for some c which is between x and x_0. Then

$$\lim_{n\to\infty} R_n(x) \le \lim_{n\to\infty} \Big|\frac{1}{(n+1)!} e^c (x-x_0)^{n+1}\Big| \le \lim_{n\to\infty} \Big|\frac{(x-x_0)^{n+1}}{(n+1)!} e^{|x|+|x_0|}\Big| = 0.$$

6.4.12. We use Taylor's Theorem to expand $f(x) = \sin x$ and $x_0 = 0$ to get

$$f(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \cos c \frac{x^7}{5040},$$

for some c between x and 0, which gives

$$\Big|\sin x - x + \frac{x^3}{6} - \frac{x^5}{120}\Big| \le \frac{1}{5040},$$

since $|x^7| \le 1$,$|\cos c| \le 1$.

6.4.14. (a) We calculate directly to get $f'(x) = 3x^2$, $f''(x) = 6x$ and $f'''(x) = 6$. Then $f(0) = f'(0) = f''(0) = 0$ and $f'''(0) > 0$. Hence it is not a relative extremum.

(b) It holds that $g'(x) = \cos x - 1$, $g''(x) = -\sin x$ and $g'''(x) = -\cos x$. Then $g(0) = g'(0) = g''(0) = 0$ and $g'''(0) < 0$. Hence, it is not a relative extremum.

(c) It holds that $h'(x) = \cos x + \frac{1}{2}x^2$. $h'(0) > 0$. Hence, it is not a relative extremum.

(d) One has $k'(x) = -\sin x + x$, $k''(x) = -\cos x + 1$, $k'''(x) = \sin x$ and $k^{(4)}(x) = \cos x$. Then $k(0) = k'(0) = k''(0) = k'''(0) = 0$ and $k^{(4)}(0) > 0$. $x = 0$ is a relative minimum point.

6.4.16. By L'Hospital's Rule,

$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = \lim_{h\to 0} \frac{f'(a+h) - f'(a-h)}{2h} = \lim_{h\to 0} (\frac{1}{2}\frac{f'(a+h) - f'(h)}{h} + \frac{1}{2}\frac{f'(h) - f'(a-h)}{h}) = f''(a).$$

Without loss of generality, we may assume $a = 0$. Let $f(x) = x^2$ for $x \geq 0$ and $f(x) = -x^2$ for $x < 0$. It is direct to see it is a counter-example.

6.4.23. We have $f'(x) = 24x^2 - 16x$. By direct calculations, one gets

(a)

$$x_1 \approx \frac{35}{52}$$
$$\cdots$$
$$x_{11} \approx 0.80901699.$$

(b)

$$x_1 = \frac{1}{2},$$

which is indeed a root. Hence, the iteration stops.