THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I

Tutorial 9

Uniform Continuity

Definition. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say that f is **uniformly continuous** on A if for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $x, u \in A$ satisfy $|x - u| < \delta(\varepsilon)$, then $|f(x) - f(u)| < \varepsilon$.

Nonuniform Continuity Criteria. *Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Then the following statements are equivalent:*

(i) f is not uniformly continuous on A.

(ii) There exists an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are points x_δ, u_δ in A such that $|x_\delta - u_\delta| < \delta$ and $|f(x_\delta) - f(u_\delta)| \geq \varepsilon_0$.

(iii) There exists an $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in A such that $\lim(x_n - u_n) = 0$ but $|f(x_n) - f(u_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

Example 1. Determine whether the following functions are uniformly continuous:

(a) $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$;

(b) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \cos(x^2)$.

Example 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If f^2 is uniformly continuous on $\mathbb{R}$, is it true that f is also uniformly continuous on $\mathbb{R}$?

Example 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function on $\mathbb{R}$ with $f(0) = 0$. Prove that there exists some $C > 0$ such that

$$|f(x)| \leq 1 + C|x| \qquad \text{for all } x \in \mathbb{R}.$$