THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I

Tutorial 7

Continuous Functions

Let $\emptyset \neq A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in A$.

Definition.

- We say that f is **continuous at** c if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $x \in A$ with $|x - c| < \delta$.
- Let $B \subseteq A$. We say that f is **continuous on** B if f is continuous at every point of B.

Remarks. (1) We do not assume that c is a limit point of A.

Case 1: If $c \in D(A)$, then f is continuous at $c \iff \lim_{x \to c} f = f(c)$.

Case 2: If $c \notin D(A)$, then $V_\delta(c) \cap A = \{c\}$ for some $\delta > 0$, so that f is automatically continuous at c.

(2) "f is continuous on B" and "$f\big|_B$ is continuous" are different.

Sequential Criterion for Continuity. *f is continuous at c if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.*

Discontinuity Criterion. *f is discontinuous at c if and only if there is a sequence (x_n) in A that converges to c but the sequence $(f(x_n))$ does not converge to $f(c)$.*

Example 1. Determine all the points of continuity of the function $g(x) \coloneqq x\lfloor x \rfloor$.

Example 2. Give an example for each of the following:

(a) A function $f : \mathbb{R} \to \mathbb{R}$ that is continuous everywhere except at one point.

(b) A function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous everywhere.

(c) A function $f : \mathbb{R} \to \mathbb{R}$ that s continuous exactly at one point.

(d) A function $f : \mathbb{R} \to \mathbb{R}$ that is continuous on $\mathbb{R} \backslash \mathbb{Q}$ but distcontinuous on $\mathbb{Q}$.

Example 3. Let $\{r_j\}_{j=1}^N$ or $\{r_j\}_{j\in\mathbb{N}}$ be an enumeration of a countable set $C \subseteq \mathbb{R}$. Define the function $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) = \sum_{j:\, r_j < x} \frac{1}{2^j}, \quad x \in \mathbb{R}.$$

Show that

(a) φ is increasing.

(b) φ is discontinuous at every $r_j \in C$.

(c) φ is continuous at every point in $\mathbb{R}\backslash C$.

Example 4. Is there a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous on $\mathbb{Q}$ but distcontinuous on $\mathbb{R}\backslash\mathbb{Q}$?