THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I

Tutorial 6

Limits of Functions

Definition. Let $A \subseteq \mathbb{R}$.

(i) A point x_0 is called a limit point of A if for any $\varepsilon > 0$, there is $a \in A$ such that $0 < |x_0 - a| < \varepsilon$.

(ii) Write $D(A)$ for the set of all limits points of A.

Definition. Let $A \subseteq \mathbb{R}$, $c \in D(A)$ and $f : A \to \mathbb{R}$. A real number L is said to be a **limit of f at c** if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever } x \in A \text{ and } 0 < |x - c| < \delta.$$

In this case, the limit is in fact unique and we write $\lim_{x \to c} f(x) = L$.

Example 1. By virtue of ε-δ definition, show that $\lim_{x \to 2} \frac{x+6}{x^2-2} = 4$.

Sequential Criterion. *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in D(A)$. Then the following are equivalent.*

(i) $\lim_{x \to c} f(x)$ exists.

(ii) If (x_n) is a convergent sequence in $A \setminus \{c\}$ with $\lim(x_n) = c$, then the sequence $(f(x_n))$ is convergent.

In this case, $\lim_{x \to c} f(x) = \lim(f(x_n))$ whenever (x_n) is a sequence in $A \setminus \{c\}$ with $\lim(x_n) = c$.

Divergence Criterion. *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in D(A)$. Then $\lim_{x \to c} f(x)$ does not exist if and only if there exists a sequence (x_n) in $A \setminus \{c\}$ convergent to c but $(f(x_n))$ does not converge in $\mathbb{R}$.*

Example 2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that $\lim_{x \to c} f(x)$ does not exist for every $c \in \mathbb{R}$.

Definition. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in D(A)$. We say that f diverges to $+\infty$ (resp. $-\infty$) as x tends to c if for any $M > 0$, there is $\delta > 0$ such that $f(x) > M$ (resp. $f(x) < -M$) whenever $x \in A$ with $0 < |x - c| < \delta$. In this case, we write $\lim_{x \to c} f(x) = +\infty$ (resp. $\lim_{x \to c} f(x) = -\infty$).

Example 3. Is there a function $f : \mathbb{R} \to \mathbb{R}$ that is not bounded in any neighbourhood at all points?

Example 4. Is there a function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x \to c} f(x) = +\infty$ for any $c \in \mathbb{R}$?