THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I

Tutorial 5

The Cauchy Criterion

Definition. A sequence $X = (x_n)$ of real numbers is said to be a **Cauchy sequence** if for any $\varepsilon > 0$ there exists a natural number K such that

$$|x_n - x_m| < \varepsilon \qquad \text{whenever } m, n \geq K.$$

Remarks. Not only difference of consecutive terms are considered!

Cauchy Convergence Criterion. *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Example 1. Let (x_n) be a sequence defined by

$$x_1 = 1 \quad \text{and} \quad x_{n+1} = \sqrt{x_n^2 + \frac{1}{2^n}} \quad \text{for } n \in \mathbb{N}.$$

Determine whether (x_n) is convergent or divergent.

Definition. A sequence (x_n) of real numbers is said to be **contractive** if there exists a constant C, $0 < C < 1$, such that

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n| \qquad \text{for all } n \in \mathbb{N}. \tag{\#}$$

The number C is called the **constant** of the contractive sequence.

Remarks. Do not confuse (#) with the following condition:

$$|x_{n+2} - x_{n+1}| < |x_{n+1} - x_n| \qquad \text{for all } n \in \mathbb{N}. \tag{\#\#}$$

For example, $(\sqrt{n})$ satisfies (##) but it is not contractive.

Theorem 1. *Every contractive sequence is a Cauchy sequence, and therefore is convergent.*

Corollary 2. *If (x_n) is a contractive sequence with constant C, $0 < C < 1$, and if $x^* := \lim(x_n)$, then*

(i) $|x^* - x_n| \leq \dfrac{C^{n-1}}{1-C}|x_2 - x_1|$,

(ii) $|x^* - x_n| \leq \dfrac{C}{1-C}|x_n - x_{n-1}|$.

Example 2. (Sequence of Fibonacci Fractions) Consider the sequence of Fibonacci fractions $x_n := f_n/f_{n+1}$, where (f_n) is the Fibonacci sequence defined by $f_1 = f_2 = 1$ and $f_{n+2} := f_{n+1} + f_n$, $n \in \mathbb{N}$. Show that the sequence (x_n) converges to $1/\varphi$, where $\varphi := (1 + \sqrt{5})/2$ is the Golden Ratio.

Example 3. The cubic equation $x^3 - 7x + 2 = 0$ has a solution between 0 and 1. Approximate this solution by means of an iteration procedure.