THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I

Tutorial 4

Definition. Let $A \subseteq \mathbb{R}$.

(i) A point x is called a limit point of A if for any $\varepsilon > 0$, there is $a \in A$ such that $0 < |x - a| < \varepsilon$.

(ii) Write $D(A)$ for the set of all limits points of A.

(iii) A is said to be closed if $D(A) \subseteq A$.

Proposition 4.7. *Let $A \subseteq \mathbb{R}$. Then the following statements are equivalent.*

(i) A is closed.

(ii) If (x_n) is a convergent sequence in A, then $\lim x_n \in A$.

Example 1. Show that

(a) if $A, B \subseteq \mathbb{R}$ are closed, then $A \cup B$ is closed;

(b) if $\{A_\alpha\}_{\alpha \in \Lambda}$ is a collection of closed subsets of $\mathbb{R}$, then $\bigcap_{\alpha \in \Lambda} A_\alpha$ is closed.

Definition. For $A \subseteq \mathbb{R}$, put

$$\overline{A} = A \cup D(A).$$

The set $\overline{A}$ is called the closure of A.

Proposition 3.10. *Let $A \subset \mathbb{R}$. Then we have the following assertions.*

1. *$\overline{A}$ is closed.*
2. *A is closed if and only if $\overline{A} = A$.*
3. *$z \in \overline{A}$ if and only if for any $\delta > 0$, there is $a \in A$ such that $|z - a| < \delta$*
   if and only if there is a convergent sequence (x_n) in A so that $z = \lim x_n$.
4. *$\overline{A}$ is the smallest closed set containing A.*

Example 2. Let $A, B \subseteq \mathbb{R}$. Show that

(a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ but $\overline{A \cap B} \not\supseteq \overline{A} \cap \overline{B}$ in general.

What can you say about infinite union?

Definition. A subset A of $\mathbb{R}$ is said to be compact if for any sequence (x_n) in A, there is a convergent subsequence (x_{n_k}) of (x_n) such that $\lim x_{n_k} \in A$.

Theorem 4.12. *Let $A \subseteq \mathbb{R}$. Then A is compact if and only if A is a closed and bounded subset.*

Example 3 (Cantor's Intersection Theorem)**.** Let $\{C_k\}_{k=1}^{\infty}$ be a sequence of non-empty, compact subsets of $\mathbb{R}$ satisfying

$$C_1 \supseteq C_2 \supseteq \cdots \supseteq C_k \supseteq C_{k+1} \supseteq \cdots .$$

Show that $\bigcap_{k=1}^{\infty} C_k \neq \emptyset$.

Definition. A set $E \subset \mathbb{R}$ is said to be perfect if E is closed and every point of E is a limit point of E.

Example 4. Let P be a non-empty perfect set. Show that P is uncountable.