THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I

Tutorial 3

Subsequences

Definition. Let (x_n) be a sequence of real numbers and let $n_1 < n_2 < \cdots < n_k < \cdots$ be a **strictly increasing** sequence of natural numbers. Then the sequence (x_{n_k}) is called a **subsequence** of (x_n).

Theorem 1. *Let (x_n) be a sequence of real numbers. Then the following are equivalent:*

(i) (x_n) *does not converge to* $x \in \mathbb{R}$.

(ii) There exists $\varepsilon_0 > 0$ *such that for any* $k \in \mathbb{N}$, *there exists* $n_k \in \mathbb{N}$ *such that* $n_k \geq k$ *and* $|x_{n_k} - x| \geq \varepsilon_0$.

(iii) There exists $\varepsilon_0 > 0$ *and a subsequence* (x_{n_k}) *of* (x_n) *such that* $|x_{n_k} - x| \geq \varepsilon_0$ *for all* $k \in \mathbb{N}$.

Example 1. Let $\ell \in \mathbb{R}$. Show that a sequence (x_n) converges to ℓ if and only if every subsequence of (x_n) has a further subsequence that converges to ℓ

Example 2. Show that if (x_n) is unbounded, then it has a subsequence (x_{n_k}) such that $\lim(1/x_{n_k}) = 0$.

Bolzano-Weierstrass Theorem

The Bolzano-Weierstrass Theorem. *A bounded sequence of real numbers has a convergent subsequence*

Example 3. Prove that a bounded divergent sequence has two subsequences converging to different limits.

Limit Superior and Limit Inferior

Let (x_n) be a bounded sequence of real numbers. For each $n \in \mathbb{N}$, define

$$a_n = \inf_{k \geq n} x_k = \inf\{x_k : k \geq n\} \quad \text{and} \quad b_n = \sup_{k \geq n} x_k = \sup\{x_k : k \geq n\}.$$

Then (a_n) and (b_n) are both monotone ((a_n) increasing and (b_n) decreasing) and bounded, hence convergent.

Definition. The **limit inferior** and **limit superior** of (x_n) are defined, respectively, by

$$\underline{\lim}\, x_n := \lim a_n = \sup_{n\geq 1}\left(\inf_{k\geq n} x_k\right),$$
$$\overline{\lim}\, x_n := \lim b_n = \inf_{n\geq 1}\left(\sup_{k\geq n} x_k\right).$$

Example 4. Alternate the terms of the sequences $(1+1/n)$ and $(-1/n)$ to obtain the sequence (x_n) given by

$$(2, -1, 3/2, -1/2, 4/3, -1/3, 5/4, -1/4, \dots).$$

Determine the values of $\overline{\lim}(x_n)$ and $\underline{\lim}(x_n)$.

Proposition 2. *Let (x_n) be a bounded sequence of real numbers. Then we have*

(i) $\underline{\lim}\, x_n \leq \overline{\lim}\, x_n$*;*

(ii) (x_n) *converges to* ℓ *if and only if* $\overline{\lim}\, x_n = \underline{\lim}\, x_n$*. In this case, we have* $\lim x_n = \underline{\lim}\, x_n = \overline{\lim}\, x_n$*.*

Proposition 3. *Let (x_n) and (y_n) be bounded sequences of real numbers. Then we have*

(i) $\overline{\lim}(-x_n) = -\underline{\lim}\, x_n$*;*

(ii) $\overline{\lim}(ax_n) = a(\overline{\lim}\, x_n)$ *and* $\underline{\lim}(ax_n) = a(\underline{\lim}\, x_n)$ *for* $a \geq 0$*;*

(iii) if $x_n \leq y_n$ *for all* n*, then* $\overline{\lim}\, x_n \leq \overline{\lim}\, y_n$ *and* $\underline{\lim}\, x_n \leq \underline{\lim}\, y_n$*;*

(iv) $\underline{\lim}\, x_n + \underline{\lim}\, y_n \leq \underline{\lim}(x_n + y_n) \leq \underline{\lim}\, x_n + \overline{\lim}\, y_n \leq \overline{\lim}(x_n + y_n) \leq \overline{\lim}\, x_n + \overline{\lim}\, y_n$*.*

Example 5. Let (x_n) be a bounded sequence of real numbers. Let $s \in \mathbb{R}$. Show that

(i) $\overline{\lim}\, x_n \leq s$ if and only if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $x_n < s + \varepsilon$ for all $n \geq N$; and

(ii) $\overline{\lim}\, x_n \geq s$ if and only if for any $\varepsilon > 0$, for all $N \in \mathbb{N}$, there is $n \geq N$ such that $x_n > s - \varepsilon$.

Example 6. Let (x_n) be a sequence of positive real numbers such that (x_{n+1}/x_n) is bounded. Show that $(\sqrt[n]{x_n})$ is also bounded and that

$$\liminf_n \frac{x_{n+1}}{x_n} \leq \liminf_n \sqrt[n]{x_n} \leq \limsup_n \sqrt[n]{x_n} \leq \limsup_n \frac{x_{n+1}}{x_n}.$$