THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I

Tutorial 2

The Limit of a Sequence

Definition. A sequence $X = (x_n)$ in $\mathbb{R}$ is said to **converge** to $x \in \mathbb{R}$, or x is said to be a **limit** of (x_n), if for every $\varepsilon > 0$ there exists a natural number $K(\varepsilon)$ such that for all $n \geq K(\varepsilon)$, the terms x_n satisfy $|x_n - x| < \varepsilon$.

Notations: $\lim X = x, \quad \lim(x_n) = x, \quad \lim_n x_n = x, \quad \lim_{n\to\infty} x_n = x.$

Steps. *To show that $\lim(x_n) = x$, we may proceed as follow:*

(1) Fix an $\varepsilon > 0$. (ε is arbitrary, but cannot be changed once fixed.)

(2) Find a useful estimate for $|x_n - x|$.

(3) Find $K(\varepsilon) \in \mathbb{N}$ such that the estimate in (2) is less than ε whenever $n \geq K(\varepsilon)$.

(4) Complete the proof.

Example 1. Use the definition of the limit of a sequence to show $\lim\left(\dfrac{n^2 - n}{2n^2 + 3}\right) = \dfrac{1}{2}$.

Example 2. Let (y_n) be a sequence of positive numbers such that $\lim_n y_n = 2$. By virtue of ε-N terminology, show that

$$\lim_n \frac{y_n}{y_n^2 - 3} = 2.$$

Example 3. Let (x_n) be a sequence of real numbers. Define

$$s_n = \frac{x_1 + x_2 \cdots + x_n}{n} \qquad \text{for all } n \in \mathbb{N}.$$

(a) If $\lim(x_n) = \ell$, where $\ell \in \mathbb{R}$, show that $\lim(s_n) = \ell$.

(b) Is the converse of (a) true?

Squeeze Theorem. *Let (x_n), (y_n) and (z_n) be sequences of real numbers such that*

$$x_n \leq y_n \leq z_n \quad \textit{for all } n \in \mathbb{N}.$$

If $\ell := \lim x_n = \lim z_n$, then (y_n) is convergent and $\lim y_n = \ell$.

Example 4. Show that $\displaystyle\lim_{n\to\infty} \frac{(2\sqrt[n]{n} - 1)^n}{n^2} = 1$.