THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I

Tutorial 1

Axiom of Completeness of $\mathbb{R}$

Definition. Let S be a nonempty subset of $\mathbb{R}$. Suppose S is bounded above. Then $u \in \mathbb{R}$ is said to be a **supremum** of S if it satisfies the conditions:

(i) u is an upper bound of S (that is, $s \leq u$ for all $s \in S$), and

(ii) if v is any upper bound of S, then $u \leq v$.

Here (ii) is equivalent to either of the following:

(ii)' if $v < u$, then there exists $s_v \in S$ such that $v < s_v$,

(ii)" for any $\varepsilon > 0$, there exists $s_0 \in S$ such that $u - \varepsilon < s_0$.

Remarks. (1) u may or may not be an element of S.

(2) The number u is unique and we write $\sup S = u$.

(3) $\inf S$ can be defined similarly provided S is bounded below.

Example 1. Find the infimum and supremum, if they exist, of the set $A := \{x \in \mathbb{R} : 1/x < x\}$. Justify your answers.

Axiom of Completeness of $\mathbb{R}$. *Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.*

Example 2. Let A and B be nonempty subsets of $\mathbb{R}$, and let $A+B := \{a+b : a \in A, b \in B\}$. Prove that, if A and B are bounded above, then

$$\sup(A+B) = \sup A + \sup B.$$

Archimedean Property. *For each $x \in \mathbb{R}$, there is a positive integer n such that $x < n$.*

Example 3. Determine the supremum and infimum of the set

$$S := \left\{ \frac{k}{2^n} : k, n \in \mathbb{N},\ \frac{k}{2^n} < \sqrt{2} \right\}.$$

Justify your answer.

Example 4. Let $\omega \in \mathbb{R}$ be an irrational positive number. Set

$$A = \{m + n\omega : m + n\omega > 0 \text{ and } m, n \in \mathbb{Z}\}.$$

Show that $\inf A = 0$.