THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I

Suggested Solution of Class Test

1. (10 *points*)

   (i) Use the $\varepsilon - \delta$ notation to see whether the limit $\lim\limits_{x\to 0^+} \dfrac{\tan x}{x^2}$ exists.

   (ii) Let $f(x) := x - [[x]]$ for $x > 0$, where $[[x]] := \min\{n \in \mathbb{N} : x < n\}$. Find the set of all discontinuous points of f.

Solution. (i) We will show that $\lim\limits_{x\to 0^+} \dfrac{\tan x}{x^2} = +\infty$.

Note that for $x > 0$, we have $\tan x \geq x$, and hence $\frac{\tan x}{x^2} \geq \frac{1}{x}$.
Let $M > 0$. Take $\delta = 1/M > 0$. Then, for any $x \in (0, \infty)$ with $0 < x - 0 < \delta$, we have

$$\frac{\tan x}{x^2} \geq \frac{1}{x} > \frac{1}{\delta} = M.$$

Hence $\lim\limits_{x\to 0^+} \dfrac{\tan x}{x^2} = +\infty$.

(ii) Note that

$$f(x) = \begin{cases} x - 1 & \text{if } \ 0 < x < 1 \\ x - n - 1 & \text{if } \ n \leq x < n + 1, \ n \in \mathbb{N}. \end{cases}$$

We will show that f is discontinuous on $\mathbb{N}$, and continuous elsewhere on $(0, \infty)$.
Let $n \in \mathbb{N}$. Then $\lim\limits_{x\to n^-} f(x) = \lim\limits_{x\to n^-} (x - n) = 0$ but $f(n) = n - n - 1 = -1$. Hence f is discontinuous at n.
For $c \in (0, \infty)\backslash\mathbb{N}$, there is $n \in \mathbb{N}$ such that $n - 1 < c < n$. Take $\delta = \frac{1}{2}\min\{c - (n - 1), n - c\}$. Then, for any $x \in (c - \delta, c + \delta)$,

$$|f(x) - f(c)| = |(x - n) - (c - n)| = |x - c|.$$

So f is continuous at c.

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2. (20 *points*) Let $f : \mathbb{R} \to [0, 1]$ be a function.

   (i) Use the $\varepsilon - \delta$ notation to show that f is discontinuous at a point $c \in \mathbb{R}$ if and only if $\lim\limits_{x\to c^+} f(x) \neq f(c)$ or $\lim\limits_{x\to c^-} f(x) \neq f(c)$.

   (ii) Show that if f is strictly increasing, then the set of all discontinuous points of f is countable.

Solution. (i) It is the same as showing that f is continuous at $c \in \mathbb{R}$ if and only if $\lim\limits_{x\to c^+} f(x) = f(c)$ and $\lim\limits_{x\to c^-} f(x) = f(c)$.

Clearly $D_r(\mathbb{R}) = D_l(\mathbb{R}) = \mathbb{R}$. So we can talk about one-sided limits at any point $c \in \mathbb{R}$.

$(\implies)$ Suppose f is continuous at $c \in \mathbb{R}$. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$. In particular, $|f(x) - f(c)| < \varepsilon$ for all x with $0 < x - c < \delta$; and $|f(x) - f(c)| < \varepsilon$ for all x with $0 < c - x < \delta$. Therefore, $\lim_{x \to c^+} f(x) = f(c)$ and $\lim_{x \to c^-} f(x) = f(c)$.

$(\impliedby)$ Suppose $\lim_{x \to c^+} f(x) = f(c)$ and $\lim_{x \to c^-} f(x) = f(c)$. Let $\varepsilon > 0$. Then there is $\delta_1 > 0$ such that $|f(x) - f(c)| < \varepsilon$ for all x with $0 < x - c < \delta_1$; and there is $\delta_2 > 0$ such that $|f(x) - f(c)| < \varepsilon$ for all x with $0 < c - x < \delta_2$. By taking $\delta = \min\{\delta_1, \delta_2\}$, we have (since the inequality below is clearly true when $x = c$),

$$|f(x) - f(c)| < \varepsilon \quad \text{whenever } |x - c| < \delta.$$

Therefore f is continuous at c.

(ii) Let D be the set of discontinuity points of f. Since f is increasing, the limits

$$f(c+) = \lim_{x \to c^+} f(x) \quad \text{and} \quad f(c-) = \lim_{x \to c^-} f(x), \quad c \in \mathbb{R}$$

exist and satisfy $f(c-) \le f(c) \le f(c+)$. So $c \in D$ if and only if $f(c) - f(c-) > 0$ or $f(c+) - f(c) > 0$. Put $J(c-) := [f(c-), f(c)]$ and $J(c+) := [f(c), f(c+)]$. Then $J(c+)$ or $J(c-)$ is an interval. Therefore, if we put $\alpha(c)$ the length of $(J(c-) \cup J(c+))$ for $c \in D_m$, then $\alpha(c) > 0$. On the other hand, if $c_1, c_2 \in D$ with $c_1 < c_2$, then $J(c_1+) \cap J(c_2-)$ has at most one point if they exist. Thus, we have

$$0 < \sum_{c \in D} \alpha(c) \le 1 - 0 = 1.$$

Since $\alpha(c) > 0$ for all $c \in D$, the set D needs to be countable. In fact, note that we have

$$D = \bigcup_{c \in \mathbb{Z}+} \{c \in D : \alpha(c) \ge 1/k\}.$$

Thus, if D is uncountable, then there exists a positive integer k so that $R := \{c \in D : \alpha(c) \ge 1/k\}$ is infinite. Therefore, $\sum_{c \in R} \alpha(c)$ is infinite. It leads to a contradiction.

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3. (20 *points*) Prove or disprove the following statements.

(i) Let $f : A \to B$ be a homeomorphism from A onto B, where A and B are non-empty subsets of $\mathbb{R}$. Let (x_n) be a sequence in A. If (x_n) is a Cauchy sequence, then so is $f(x_n)$.

(ii) If $f : (0, 1) \to \mathbb{R}$ is a bounded continuous injection, then it is impossible to find a point $x_0 \in (0, 1)$ such that $f(x_0) = \sup\{f(x) : x \in (0, 1)\}$.

(iii) Let $f : E \to \mathbb{R}$ be a continuous function defined on a closed subset E of $\mathbb{R}$. If z is a limit point of $f(E)$, then $z \in f(E)$.

Solution. (i) The statement is false.

Consider the function $f : (0, 1] \to [1, \infty)$ defined by $f(x) = 1/x$. Then f is a continuous bijection with inverse $f^{-1}(x) = 1/x$, which is also continuous. So f is a homeomorphism.

For $x_n := 1/n$, (x_n) is convergent, hence a Cauchy sequence in $(0, 1)$. However, $f(x_n) = n$ is unbounded. So $(f(x_n))$ is not a Cauchy sequence.

(ii) The statement is true.

Suppose there is $x_0 \in (0, 1)$ such that $f(x_0) = \sup\{f(x) : x \in (0, 1)\}$. Since $x_0 \in (0, 1)$, there exist $x_1, x_2 \in (0, 1)$ such that $x_1 < x_0 < x_2$. Because f is injective and $f(x_0) = \sup\{f(x) : x \in (0, 1)\}$, we have either

$$f(x_1) < f(x_2) < f(x_0)$$

or

$$f(x_2) < f(x_1) < f(x_0).$$

If it is the first case, then the Intermediate Value Theorem implies that there is $c \in (x_1, x_0)$ such that $f(c) = f(x_2)$, contradicting the injectivity of f. If it is the second case, the same argument leads to a contradiction.

(iii) The statement is false.

Consider the function $f : [1, \infty) \to \mathbb{R}$ defined by $f(x) = 1/x$. Then $[1, \infty)$ is a closed subset of $\mathbb{R}$, f is a continuous function on $[1, \infty)$, and $f([1, \infty)) = (0, 1]$. Now 0 is a limit point of $(0, 1]$ but $0 \notin (0, 1]$.

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