Math2058 Mathematical Analysis (I): 2023-24

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Abstract

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1 Order structure of $\mathbb{R}$

Throughout this section, a number L means that it is a real number and let S be a non-empty subset of $\mathbb{R}$.

Definition 1.1 *Using the notation as above:*

(i) S is said to be bounded above (resp. bounded below) if there is a number L (resp. ℓ such that $x \leq L$ (resp. $\ell \leq x$) for all $x \in S$. In this case, such number L (resp. ℓ) is called an upper bound (resp. lower bound) for S.
Furthermore, S is said to be bounded if it is both are bounded above and bounded below.

(ii) S is said to have a maximal element (resp. minimal element) if there is an element $M \in S$ (resp. $m \in S$ such that $x \leq M$ (resp. $m \leq x$) for all $x \in S$. In this case, write $\max S$ and $\min S$ for the maximal element and the minimal element of S respectively.

Remark 1.2 (i) It is noted that the maximum of a set may not exist even it is bounded above. For example, if let $S = \{1 - \frac{1}{n} : n = 1, 2...\}$, then S is bounded above but $\max S$ does not exist.

(ii) It is clear that $\max S$ exists if and only if $\min(-S)$ exists, where $-S = \{-x : x \in S\}$. In this case, we have $-\max S = \min(-S)$.

The following notion plays an important role in mathematics.

Definition 1.3 *Using the notation as above, a number $L \in \mathbb{R}$ (rep. ℓ) is called the* **supremum** *(resp. the* **infimum** *) of S if L is the least upper bound (resp. the greatest lower bound) for S. In this case, we write*

$$L := \sup S \qquad ; \qquad \ell := \inf S.$$

The following result is easy shown by the fact that a number L is an upper bound for S if and only if $-L$ is a lower bound for the set $-S$.

Proposition 1.4 *Using the notation as above, then* $\sup S$ *exists if and only if* $\inf(-S)$ *exists. In this case, we have*

$$-\sup S = \inf(-S).$$

The following is a very useful result for checking a number whether it is the supremum of a given set. In addition, the technique of the proof is standard.

Theorem 1.5 *Assume that* $\sup S$ *exists. A number* $L = \sup S$ *if and only if it satisfies the following two conditions.*

(i) L *is an upper bound for* S*.*

(ii) For any $\varepsilon > 0$*, there is an element* $x_0 \in S$ *such that* $L - \varepsilon < x_0$*.*

Similarly, if $\inf S$ *exists, then a number* $\ell = \inf S$ *if and only if the following two conditions hold:*

(i') ℓ *is a lower bound for* S*;*

(ii') For any $\varepsilon > 0$*, there is an element* $y_0 \in S$ *such that* $y_0 < \ell + \varepsilon$*.*

Proof: We are going to show the case of supremum first. For showing ($\Rightarrow$), assume that $L = \sup S$. It is noted that the condition (i) automatically holds by the definition of supremum. It remains to show that the condition (ii) holds. Let $\varepsilon > 0$. Then $L - \varepsilon < L$. Since L is the least upper bound for S, $L - \varepsilon$ is not an upper bound for S. Therefore, there is an element $X_0 \in S$ such that $L - \varepsilon < x_0$ as desired.
Now for showing the converse statement, assume that the conditions (i) and (ii) hold for the number L. Then by the definition of the supremum, it needs to show that if L_1 is an upper bound for S, then $L \leq L_1$. Suppose not, that is, we assume that there is an upper bound L_1 for S such that $L_1 < L$. Then $\varepsilon := 1/2(L - L_1) > 0$. The condition (ii) gives an element $x_0 \in S$ such that

$$L_1 < \frac{1}{2}(L_1 + L) = L - \varepsilon < x_0 \leq L.$$

The last statement can be obtained by considering $-S$ in the first assertion above. □

Axiom of Completeness of $\mathbb{R}$: *Every bounded above non-empty subset of* $\mathbb{R}$ *must have the least upper bound, that is, the supremum of a bounded above non-empty subset of* $\mathbb{R}$ *must exist.*

Proposition 1.6 *Let* A *and* B *be non-empty bounded above subsets of* $\mathbb{R}$*. Put* $A + B := \{x + y : x \in A, y \in B\}$*. Then we have* $\sup(A + B) = \sup A + \sup B$*.*

Proof: Note that $L_1 := \sup A$ and $L_2 := \sup B$ exist by the Axiom of Completeness. It is clear that $L_1 + L_2$ is an upper bound for the set $A + B$. By using Theorem 1.5, it suffices to show the condition (ii) in Theorem 1.5 holds. Let $\varepsilon > 0$. Then by Theorem 1.5, there are elements $a \in A$ and $b \in B$ such that $L_1 - \frac{1}{2}\varepsilon < a$ and $L_2 - \frac{1}{2}\varepsilon < b$. Hence, we have $L_1 + L_2 - \varepsilon < a + b$. Thus the condition (ii) holds for the set $A + B$. The proof is finished.
□

Proposition 1.7 *If S is a bounded below non-empty subset of $\mathbb{R}$, then $\inf S$ must exist.*

Proof: Note that the set $-S$ is bounded above. Then by the completeness of $\mathbb{R}$, $\sup(-S)$ exists and hence, $\inf S = -\sup(-S)$ must exist. □

Theorem 1.8 Archimedean Property: *For each $x \in \mathbb{R}$, there is a positive integer n such that $x < n$.*

Proof: The proof is shown by the contradiction. Suppose that there is a real number M such that $n \leq M$ for all $n \in \mathbb{Z}_+$. Thus, the set of all positive integers $\mathbb{Z}_+$ is bounded above. The Axiom of Completeness tells us that the supremum $L := \sup \mathbb{Z}_+$ must exist. Then by considering $\varepsilon = 1$ in Theorem 1.5, there is an element $m \in \mathbb{Z}_+$ such that $L - 1 < m$ and hence, $L < m + 1$. This implies that $n < m + 1$ for all $n \in \mathbb{Z}_+$. It leads to a contradiction because $m + 1 \in \mathbb{Z}_+$. □

Corollary 1.9 $\inf\{1/n : n = 1, 2...\} = 0$.

Proof: Let $S := \{1/n : n = 1, 2...\}$. It is noted that 0 is a lower bound for the set S. By using Theorem 1.5, it needs to show that for any $\varepsilon > 0$, there is an element $a \in S$ such that $a < 0 + \varepsilon$. Now let $\varepsilon > 0$. Then by Archimedean property, there is a positive integer N such that $1/\varepsilon < N$. Thus, we have $1/N \in S$ and $1/N < \varepsilon$ as required. The proof is finished. □

Definition 1.10 *We say that a subset A of $\mathbb{R}$ is dense in $\mathbb{R}$ if $(a, b) \cap A \neq \emptyset$.*

Example 1.11 *The set of all integers $\mathbb{Z}$ is not dense in $\mathbb{R}$.*

The following shows that the set of rational numbers a dense subset of $\mathbb{R}$ which $\mathbb{Q}$ is an important dense subset.

Proposition 1.12 *For each pair of real numbers a and b with $a < b$, then we have $(a, b) \cap \mathbb{Q} \neq \emptyset$. In this case, the set of all rational numbers is dense in $\mathbb{R}$.*

Proof: Note that we may assume $0 < a < b$. (Why?) By using Corollary 1.9, there is a positive integer N such that $1/N < b - a$ and hence, we have $1 < Nb - Na$. On the other hand, let $p := \max\{k \in \mathbb{N} : k \leq Na\}$. This implies that $Na < p + 1$. In addition, since $Nb - Na > 1$, we have $p + 1 < Nb$. Therefore, we have $Na < p + 1 < Nb$. Thus, $\frac{p+1}{N} \in \mathbb{Q} \cap (a, b)$. The proof is finished. □

Before showing the following proposition, we have a simple but useful observation first.

Lemma 1.13 *Let $e, f \in \mathbb{R}$. Then we have $e \leq f$ if and only if for all $\varepsilon > 0$, we have $e < f + \varepsilon$.*

Proposition 1.14 *There is a unique real number x such that $x^2 = 2$. Consequently, such real number is irrational.*

Proof: Let $S := \{x > 0 : x^2 \leq 2\}$. Note that $1 \in S$, hence, $S \neq \emptyset$. On the other hand, if $x > 2$, then $x^2 > 4$. This implies that the set S is bounded by 2 and thus, the set S is bounded above. Then the Axiom of Completeness assures that $a := \sup S$ exists. We are going to show that $a^2 = 2$ as required.
We first note that by the characterization of the sup, for each positive integer n, there is an element $x_n \in S$ such that $a - \frac{1}{n} < x_n$. This implies that

$$a^2 < (x_n + \frac{1}{n})^2 = x_n^2 + \frac{2}{n}x_n + \frac{1}{n^2} < 2 + \frac{4}{n} + \frac{1}{n^2} \tag{1.1}$$

It is noted that we have $\frac{4}{n}+\frac{1}{n^2} < \frac{5}{n}$ for all positive integer n. Therefore, we have $\inf\{\frac{4}{n}+\frac{1}{n^2} : n = 1, 2....\} = 0$ because $\inf\{1/n : n =, 1, 2....\} = 0$. This implies that for any $\varepsilon > 0$, there is a positive integer m such that $\frac{4}{m} + \frac{1}{m^2} < \varepsilon$. Therefore, we have

$$a^2 < 2 + \epsilon$$

for all $\varepsilon > 0$. Lemma 1.13 implies that $a^2 \leq 2$.
Finally, it remains to show that $a^2 < 2$ is impossible. Assume that $a^2 < 2$. Then by using the fact $\inf\{\frac{4}{n} + \frac{1}{n^2} : n = 1, 2....\} = 0$ again, one can choose a positive integer N such that $\frac{4}{N} + \frac{1}{N^2} < 2 - a^2$, and hence, we have $a^2 + \frac{4}{N} + \frac{1}{N^2} < 2$. This implies that

$$(a + 1/N)^2 = a^2 + \frac{2}{N}a + \frac{1}{N^2} \leq a^2 + \frac{4}{N} + \frac{1}{N^2} < 2.$$

Thus, we have $(a + 1/N) \in S$ and $a < a + 1/N$. It leads to a contradiction. Therefore, $a^2 = 2$.
The uniqueness clearly follows from the fact that if $a^2 = b^2 = 2$, then we have $a^2 - b^2 = (a - b)(a + b) = 0$.
Now write $\sqrt{2} := \sup S$. Then by above we have $(\sqrt{2})^2 = 2$. Suppose that $\sqrt{2} = p/q$ is rational, for some positive integers p and q. We have $p^2 = 2q^2$. Then by the Unique Prime Factorization theorem, there are natural numbers n and s such that $p = 2^n s$ and s is not divided by 2. Similarly, there are natural numbers m and t such that $q = 2^m t$ and t is not divided by 2. Thus, we have

$$2^{2n}s^2 = p^2 = 2q^2 = 2 \cdot 2^{2m}t^2 = 2^{2m+1}t^2.$$

From this we have $2n = 2m + 1$. It is impossible. The proof is complete. □

Theorem 1.15 *For any open interval* (a, b)*, we have* $(a, b) \cap \mathbb{Q}^c \neq \emptyset$*, i.e., the set of all irrational numbers is dense in* $\mathbb{R}$*.*

Proof: We may assume that $a > 0$ and hence, we have $\sqrt{2}a < \sqrt{2}b$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is an element $r \in \mathbb{Q} \cap (\sqrt{2}a, \sqrt{2}b)$. Hence, we have

$$a < \frac{r}{\sqrt{2}} < b.$$

Since $\sqrt{2}$ is irrational and r is rational, we see that the number $\frac{r}{\sqrt{2}}$ is irrational as required.
□

2 Sequences

A *sequence* of real numbers means that it is a real-valued function x defined on $\mathbb{Z}_+$ (or $\mathbb{N}$). Write $x_n := x(n)$ for $n = 1, 2...$ and $x = (x_n)$.

The following definition plays a very important role in mathematics.

Definition 2.1 *We say that a sequence (x_n) is convergent if there is a number $L \in \mathbb{R}$ which satisfies the following condition:*
For any $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ (depends on the choice of ε), such that

$$|x_n - L| < \varepsilon \quad \text{whenever } n \geq N.$$

In this case, we say that (x_n) converges to L and L is a **limit** *of (x_n). If such L does not exist, we say that (x_n) is divergent.*

Remark 2.2 Using the notation above, we have:

(i) *A number ℓ is* **Not** *a limit of (x_n) if there is $\varepsilon > 0$ such that for any positive integer N, we can find a positive integer n with $n \geq N$ so that $|x_n - \ell| \geq \varepsilon$.*
Warning*: in this case, it does not imply that (x_n) is divergent !!!!*

(ii) The Definition 2.1 is clearly equivalent to the following statement:
there is a constant $C > 0$ such that for any $\eta > 0$, there is a positive integer N satisfying $|x_n - L| < C\eta$ as $n \geq N$.

The following is one of important properties of limits.

Proposition 2.3 *If (x_n) is a convergent sequence, then its limit is unique.*
In this case, we write $\lim x_n$ for **"the"** *limit of (x_n).*

Proof: Let L and L' be limits of (x_n). Then for any $\varepsilon > 0$, there are positive integers N and N' such that $|x_n - L| < \varepsilon$ for any $n \geq N$ and $|x_n - L'|$ for any $n \geq N'$. Now if we choose a positive m so that $m \geq N$ and $m \geq N'$, then we have

$$|L - L'| \leq |L - x_m| + |x_m - L'| < 2\varepsilon.$$

Therefore, we have $|L - L'| < 2\varepsilon$ for all $\varepsilon > 0$. This implies that $|L - L'| = 0$ and thus, $L = L'$. Otherwise, if we choose $0 < \varepsilon < \frac{1}{4}|L - L'|$, then it leads to a contradiction. $\square$

Example 2.4 *Show that if $x_n := \frac{n+1}{n-1}$ for $n = 2, 3..$, then the sequence $\lim x_n = 1$.*

Proof: Note that for each positive integer n with $n \geq 2$, we have $|x_n - 1| = \frac{2}{n-1}$. Now let $\varepsilon > 0$. Therefore, we have $|x_n - 1| < \varepsilon$ if and only if $\frac{2}{\varepsilon} + 1 < n$. The Archimedean property tells us that there is a positive integer N such that $N > \frac{2}{\varepsilon} + 1$. Hence, we have $|x_n - 1| < \varepsilon$ as $n \geq N$. The proof is complete. $\square$

Example 2.5 *Let $x_n = (-1)^n$ for $n = 1, 2....$ Show that the sequence (x_n) is divergent.*

Proof: **Warning**: It is clear that neither 1 nor -1 both is the limit of the sequence of (x_n). However, we cannot conclude from the Definition 2.1 that the sequence (x_n) is divergent since the sequence (x_n) may converge to the number which is other than 1 and -1.
Now suppose that the sequence (x_n) is convergent with $L := \lim x_n$. Now if for each positive integer N, put $A_N := \{x_n : n \geq N\}$, then $A_N = \{1, -1\}$. Therefore, for any positive integer N the intersection $(L - 1/4, L + 1/4) \cap A_N$ contains at most one point. This implies that for any positive integer N, there is $m \geq N$ such that $x_m \notin (L - 1/4, L + 1/4)$, that is, $|x_m - L| \geq 1/4$. It leads to a contradiction since L is the limit of (x_n) by the assumption. □

Example 2.6 *Show that if $x_n = n$ for all $n = 1, 2...$, then the sequence (x_n) is divergent.*

Proof: suppose not, we assume that the sequence (x_n) converges to some number L. Then by Definition 2.1, if we consider $\varepsilon = 1$, then there is a positive integer N such that $|x_n - L| < 1$ for all $n \geq N$ and thus, $n < |L| + 1$ for all $n \geq N$. This implies that $n < |L| + 1$ for all positive integers n. This contradicts to the Archimedean property. □

Using the similar idea as the proof of Example 2.6, one can obtain a more general result as follows.

Proposition 2.7 *Every convergent sequence is bounded.*

Proof: Let (x_n) be a convergent sequence with the limit L. If we take $\varepsilon = 1$ in the Definition 2.1, there is a positive integer N such that $|x_n - L| < 1$ for all $n \geq N$. Hence, we have $|x_n| < |L| + 1$ for all $n \geq N$. Thus, if we take $M := \max\{|x_1|, ..., |x_{N-1}|, |L| + 1\}$, then we have $|x_n| \leq M$ for all $n = 1, 2, ...$ Thus, (x_n) is bounded. □

Proposition 2.8 *Let (x_n) and (y_n) be the convergent sequences. Let $a := \lim x_n$ and $b := \lim y_n$. We have the following assertions.*

(i) *$(x_n + y_n)$ is convergent with $\lim(x_n + y_n) = a + b$.*

(ii) *The product $(x_n y_n)$ is convergent with $\lim x_n y_n = ab$.*

(iii) *If $y_n \neq 0$ for all n and $b \neq 0$, then the sequence (x_n/y_n) is convergent and $\lim x_n/y_m = a/b$.*

Proof: For showing (i): let $\varepsilon > 0$. Then there is a positive integer N such that $|x_n - a| < \varepsilon$ and $|y_n - b| < \varepsilon$ for all $n \geq N$. This implies that

$$|(x_n + y_n) - (a + b)| \leq |x_n - a| + |y_n - b| < 2\varepsilon$$

for all $n \geq N$. Thus, $(x_n + y_n)$ is convergent with $\lim(x_n + y_n) = a + b$.
For (ii), let $\varepsilon > 0$ and let N be chosen as in Part (i). Since (y_n) is convergent, (y_n) is bounded and hence, there is $M > 0$ such that $|y_n| \leq M$ for all n. Hence, the triangle inequality implies that

$$|x_n y_n - ab| \leq |x_n y_n - a y_n| + |a y_n - ab| \leq |x_n - a||y_n| + |a||y_n - b| \leq (M + |a|)\varepsilon$$

for all $n \geq N$. This implies that $(x_n y_n)$ is convergent and $\lim x_n y_n = ab$.
For showing (iii), it suffices to show that the sequence $(\frac{1}{y_n})$ converges to $1/b$ by using Part (ii).
Let $\varepsilon > 0$ and N be as in Part (i) again. It is noted that since $b \neq 0$, by using the Definition 2.1 there is a positive integer $N_1 > N$ such that $|y_n - b| < \frac{|b|}{2}$ for all $n \geq N_1$. This gives $|y_n| > \frac{|b|}{2}$ for all $n \geq N_1$. Hence, we have

$$|\frac{1}{y_n} - \frac{1}{b}| = \frac{|y_n - b|}{|y_n||b|} \leq \frac{2}{|b|^2}\varepsilon$$

for all $n \geq N_1$. The proof is complete. □

Proposition 2.9 *Let (x_n) and (y_n) be the convergent sequences with the limits $a := \lim x_n$ and $b := \lim y_n$. If $x_n \leq y_n$ for all $n = 1, 2...$, then $a \leq b$.*

Proof: It suffices to show that $a < b + \varepsilon$ for all $\varepsilon > 0$; otherwise, if $b < a$, then by taking $\varepsilon = a - b > 0$ we have $a < b + (a - b) = a$ which is impossible. Now let $\varepsilon > 0$. Then there is a positive integer N such that $|x_N - a| < \varepsilon$ and $|y_N - b| < \varepsilon$. This implies that

$$a - \varepsilon < x_N \leq y_N < b + \varepsilon.$$

Thus, we have $a < b + 2\varepsilon$. The proof is complete. □

Proposition 2.10 *Let $(x_n), (y_n)$ and (z_n) be the sequences which satisfy $x_n \leq y_n \leq z_n$ for all n. If $a := \lim x_n = \lim z_n$, then (y_n) is convergent and $\lim y_n = a$.*

Proof: Let $\varepsilon > 0$. Then by the Definition 2.1, there is a positive integer N such that $|x_n - a| < \varepsilon$ and $|z_n - a| < \varepsilon$ for all $n \geq N$. This implies that

$$a - \varepsilon < x_n \leq y_n \leq z_n < a + \varepsilon$$

for all $n \geq N$. Hence, we have $|y_n - a| < \varepsilon$ for all $n \geq N$. The proof is finished. □

Proposition 2.11 *let S be a non-empty bounded above subset of $\mathbb{R}$. Then a number $L = \sup S$ if and only if L is an upper bound for S and there is a sequence (x_n) in S such that $\lim x_n = L$.*

Proof: For showing $(\Rightarrow)$, assume $L = \sup S$. Then L is an upper bound for S by the definition. It suffices to show that there is a sequence (x_n) in S such that $\lim x_n = L$. Recall the characterization of supremum that for any $\varepsilon > 0$, there is an element $x \in S$ such that $L - \varepsilon < x$. From this for each positive integer n, there is an element $x_n \in S$ such that $L - \frac{1}{n} < x_n \leq L$. This implies that $|x_n - L| < \frac{1}{n}$ for all n and thus, $\lim x_n = L$ as required. The converse is clear due to the characterization of supremum again. □

Definition 2.12 *A sequence (x_n) is said to be increasing (resp. decreasing) if $x_n \leq x_{n+1}$ (resp. $x_n \geq x_{n+1}$ for all n.*

Theorem 2.13 *Let (x_n) be an increasing (resp. decreasing) sequence. Then (x_n) is convergent if and only if (x_n) is bounded. In this case, we have $\lim x_n = \sup\{x_n : n = 1, 2...\}$ (resp. $\lim x_n = \inf\{x_n : n = 1, 2...\}$).*

Proof: Assume that (x_n) is increasing. It is noted that this part ($\Rightarrow$) is always true even (x_n) is not increasing.
Now for showing the part ($\Leftarrow$), assume that (x_n) is bounded. Then the set $S := \{x_n : n = 1, 2, ...\}$ is bounded. The Axiom of Completeness tells us that $L := \sup(S)$ exists. We are going to show that $\lim x_n = L$. In fact, for any $\varepsilon > 0$, there is an element $x_N \in S$ such that $L - \varepsilon < x_N$ because $L = \sup(S)$. Since (x_n) is increasing, we have $L - \varepsilon < x_N \leq x_n \leq L$ for all $n \geq N$. Hence, $|x_n - L| < \varepsilon$ for all $n \geq N$. Therefore, (x_n) converges to L as desired.
When (x_n) is decreasing, the assertion can be obtained by considering the sequence $(-x_n)$.
□

Example 2.14 *Then the following limit exists*

$$e := \lim(1 + \frac{1}{n})^n.$$

Proof: *For each positive integer, let*

$$x_n = (1 + \frac{1}{n})^n.$$

They by using Proposition 2.13, it suffices to show that (x_n) is a bounded increasing sequence.
We first claim that (x_n) is increasing. In fact, by the Binomial Theorem, we see that

$$x_n = 1 + 1 + \sum_{k=2}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!}\frac{1}{n^k} = \sum_{k=1}^{n} \frac{1}{k!}(1 - \frac{1}{n})(1 - \frac{2}{n})\cdots(1 - \frac{k-1}{n}). \quad (2.1)$$

It is noted that each term in above is positive and the coefficients of $\frac{1}{k!}$ for $2 \leq k \leq n$ in x_n and x_{n+1} are

$$(1 - \frac{1}{n})(1 - \frac{2}{n})\cdots(1 - \frac{k-1}{n}) \quad \textit{and} \quad (1 - \frac{1}{n+1})(1 - \frac{2}{n+1})\cdots(1 - \frac{k-1}{n+1})$$

respectively. From this we see that $x_n \leq x_{n+1}$ for all n and thus, the sequence (x_n) is increasing.
It remains to show that (x_n) is bounded. In fact, for each $2 \leq k \leq n$ we have

$$\frac{1}{k!}(1 - \frac{1}{n})(1 - \frac{2}{n})\cdots(1 - \frac{k-1}{n}) < \frac{1}{2^k}.$$

Then

$$x_n < 1 + 1 + \sum_{k=1}^{n} \frac{1}{2^k} < 3.$$

The proof is complete. □

Remark 2.15 The limit e in Example 2.14 above is very important in mathematics which is called the natural base today. It was first studied by Jacob Bernoulli (1683). The notation e was introduced induced by Euler (1748). The study of the limit $\lim_n(1+\frac{1}{n})^n$ is motivated by the *Compound interest formula*, that is

$$A = P(1 + \frac{r}{n})^n$$

where A is the total amount; P is the principal; r is the annual interest rate; n is the number of payment in a year.

Theorem 2.16 Nested Intervals Theorem *Let $(I_n := [a_n, b_n])$ be a sequence of closed and bounded intervals. Assume that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$. Then we have $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Furthermore, if we further assume that $\lim_n(b_n - a_n) = 0$, then there is a unique real number c such that $\bigcap_{n=1}^{\infty} I_n = \{c\}$.*

Proof: It is noted that since (I_n) is a decreasing sequence of closed and bounded intervals, we have

$$a_1 \leq a_2 \leq \cdots \leq a_n < b_n \leq b_{n-1} \leq \cdots \leq b_2 \leq b_1$$

for all positive integers n. Therefore, (a_n) and (b_n) are bounded and they are increasing and decreasing and respectively. This implies that (x_n) and (y_n) both are convergent and $a := \lim a_n = \sup\{a_n : n = 1, 2...\}$ and $b := \lim b_n = \inf\{b_n : n = 1, 2...\}$. In addition, we have $a \leq b$ because $a_n \leq b_n$ for all n. Thus, if we fix some c such that $a \leq c \leq b$, then $c \in \bigcap_{n=1}^{\infty} I_n$ as desired because we have $a_n \leq a \leq c \leq b \leq b_n$ for all n.
It remains to show $\bigcap_{n=1}^{\infty} I_n = \{c\}$ if $\lim(b_n - a_n) = 0$. In fact,if c c' in $\bigcap_{n=1}^{\infty} I_n$, then we have $|c - c'| \leq |b_n - a_n|$ for all n. This implies that $|c - c'| = 0$ and thus, $c = c'$. The proof is finished. $\square$

Remark 2.17 The assumption of the boundedness and closeness of the intervals I_n cannot be removed in the Nest Intervals Theorem.
For example, if $I_n := (0, \frac{1}{n})$ and $J_n := [n, \infty)$, for all $n = 1, 2, ...$, then $\bigcap I_n = \bigcap J_n = \emptyset$.

3 Subsequences

Definition 3.1 *A subsequence $(x_{n_k})_{k=1}^{\infty}$ of a sequence (x_n) means that $(n_k)_{k=1}^{\infty}$ is a sequence of positive integers satisfying $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$, that is, such sequence (n_k) can be viewed as a strictly increasing function $\mathbf{n} : k \in \{1, 2, ..\} \mapsto n_k \in \{1, 2, ...\}$.*

Remark 3.2 In this case, note that for each positive integer N, there is $K \in \mathbb{N}$ such that $n_K \geq N$ and thus we have $n_k \geq N$ for all $k \geq K$.

Proposition 3.3 *If (x_n) is a convergent sequence, then any subsequence (x_{n_k}) of (x_n) converges to the same limit. In this case, we have $\lim_k x_{n_k} = \lim x_n$.*

Proof: We assume that $\lim x_n = a \in \mathbb{R}$ exists. Let (x_{n_k}) be a subsequence of (x_n). We claim that $\lim x_{n_k} = a$. Let $\varepsilon > 0$. In fact, since $\lim x_n = a$, there is a positive integer N such that $|a - x_n| < \varepsilon$ for all $n \geq N$. Note that by the definition of a subsequence, there is a positive integer K such that $n_k \geq N$ for all $k \geq K$. Hence, we see that $|a - x_{n_k}| < \varepsilon$ for all $k \geq K$. Thus we have $\lim_{k\to\infty} x_{n_k} = a$. The proof is complete. $\square$

Theorem 3.4 Bolzano-Weierstrass Theorem *(write B-W Theorem for short): Every bounded sequence has a convergent subsequence.*

Proof: We give two different proofs in here, however, each proof basically is due to the Axiom of Completeness.
Let (x_n) be a bounded sequence and put $X := \{x_n : n = 1, 2, ..\}$. The Theorem clearly holds if X is a finite set. In fact in this case, there must have an element x_m appears infinite many times. Hence, we can choose a subsequence (x_{n_k}) so that $x_{n_k} \equiv x_m$ for all $k = 1, 2....$
Thus we may assume that the set X is infinite.
Method 1:
Since (x_n) is bounded, there is a closed and bounded interval $I_1 = [a_1, b_1]$ such that $x_n \in I_1$ for all n. Put $x_{n_1} := x_1$.
It is noted that one of the following sets must be infinite:

$$A_2 := \{n \in \mathbb{Z}_+ : x_n \in [a_1, \frac{a_1 + b_1}{2}]\}; \quad B_2 := \{n \in \mathbb{Z}_+ : x_n \in [\frac{a_1 + b_1}{2}, b_1]\}.$$

We may assume that the set A_2 is infinite. Hence there is an element $n_2 \in A_2$ such that $n_1 < n_2$. Put $I_2 := [a_2, b_2] = [a_1, \frac{a_1+b_1}{2}]$. Thus $x_{n_2} \in I_2$. Similarly, one of the following sets is infinite:

$$A_3 := \{n \in \mathbb{Z}_+ : x_n \in [a_2, \frac{a_2 + b_2}{2}]\}; \quad B_3 := \{n \in \mathbb{Z}_+ : x_n \in [\frac{a_2 + b_2}{2}, b_2]\}.$$

In addition, we may assume that the set A_3 is infinite. Hence, there is an element $n_3 \in A_3$ such that $n_1 < n_2 < n_3$. Put $I_3 := [a_3, b_3] = [a_2, \frac{a_2+b_2}{2}]$. Thus, $x_{n_3} \in I_3$. By repeating the same step, we can get a decreasing sequence of a closed and bounded intervals $I_k = [a_k, b_k]$ and a subsequence (x_{n_k}) of (x_n) such that the following conditions hold:

1. $\lim(b_k - a_k) = \lim \frac{1}{2^k}(b_1 - a_1) = 0$.
2. $x_{n_k} \in I_k$ for all $k = 1, 2....$

The Nest Intervals Theorem tells us that there is a number c such that $c \in I_k$ for all k and hence, we have $|x_{n_k} - c| \leq (b_k - a_k) = \frac{1}{2^k}(b_1 - a_1) \to 0$. Therefore the subsequence (x_{n_k}) is convergent as required. The proof is finished.
Method 2
This method is the Weierstrass' original proof.
Recall our assumption that the set $X = \{x_n : n = 1, 2...\}$ is infinite. Let

$$S := \{x \in \mathbb{R} : \ (x, \infty) \cap X \text{ is infinite}\}.$$

We first note that since (x_n) is bounded, there are real numbers m and M so that $m \leq x_n \leq M$ for all n. Since the set $X = \{x_n : n = 1, 2...\}$ is infinite, the set S is a bounded above non-empty set because $m \in S$ and $x \leq M$ for all $x \in S$. The Axiom of Completeness implies that $L := \sup(S)$ must exist. We want to show that there is a subsequence (x_{n_k}) of (x_n) which converges to L.
Claim: For any $\varepsilon > 0$, there is an element $u \in S$ such that $|u - L| < \varepsilon$ and $(u, L + \varepsilon] \cap X$ is infinite.
In fact, if let $\varepsilon > 0$, then by the characterization of the supremum there is an element $u \in S$ such that $L - \varepsilon < u$. Since $u \in S$, we have $(u, \infty) \cap X$ is infinite. It implies that the set $(u, L + \varepsilon] \cap X$ must be infinite, otherwise, $(L + \varepsilon, \infty) \cap X$ is infinite and thus, $L + \varepsilon \in S$ by the construction of S. It leads to a contradiction because L is an upper bound for S. Thus,

the **Claim** follows.
Now for $\varepsilon = 1$, then there is $u_1 \in S$ such that $L-1 < u_1 < L+1$. Then by the Claim above, choose $x_{n_1} \in (u_1, L+1]$ and hence, $L-1 < x_{n_1} \leq L+1$. Next, we considering $\varepsilon = 1/2$,then there is an element $u_2 \in S$ such that the set $(u_2, L+1/2]$ is infinite by the Claim above again. Therefore we can find x_{n_2} such that $n_1 < n_2$ and $L - 1/2 < u_2 < x_{n_2} \leq L + 1/2$. By repeating the same step and considering $\varepsilon = \frac{1}{k}$ for $k = 1, 2...$ in the **Claim** above, we can get a subsequence (x_{n_k}) of (x_n) such that $L - \frac{1}{k} \leq x_{n_k} \leq L + \frac{1}{k}$ for all $k = 1, 2....$ Therefore, (x_{n_k}) is a convergent subsequence of (x_n) with the limit L. The proof is complete. □

Remark 3.5 The assumption of the boundedness of (x_n) cannot be removed. For example, let $x_n = n$ for all $n = 1, 2....$ Then (x_n) does not have a convergent subsequence because $|x_n - x_m| \geq 1$ for $n \neq m$.

Proposition 3.6 *Let (x_n) be a bounded sequence. For each positive integer n, put*

$$a_n := \inf\{x_k : k \geq n\} \quad \textit{and} \quad b_n := \sup\{x_k : k \geq n\}.$$

Then we have the following assertions.

(i) The limits $\lim a_n$ and $\lim b_n$ always exist with $\lim a_n \leq \lim b_n$. In this case, we write $\underline{\lim} x_n := \lim a_n$ (called the $\liminf$ of (x_n)) and $\overline{\lim} x_n = \lim b_n$ (called the $\limsup$ of (x_n)).

(ii) (x_n) is convergent if and only if $\underline{\lim} x_n = \overline{\lim} x_n$. In this case, we have $\lim x_n = \underline{\lim} x_n = \overline{\lim} x_n$.

(iii) There exists a subsequence (x_{n_k}) of (x_n) such that $\lim x_{n_k} = \overline{\lim} x_n$. Consequently, the Bolzano-Weierstrass Theorem holds.

Proof: For showing part (i), we note that if $a_n \leq x_k$ for all $k \geq n$, then $a_n \leq x_k$ for $k \geq n+1$. Thus, we have $a_n \leq a_{n+1}$ for all n. Similarly, we have $b_{n+1} \geq b_n$. Thus, we have $a_1 \leq \cdots a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq \cdots \leq b_1$ for all n. This implies that (a_n) and b_n both are bounded monotone sequences. Therefore, $\lim a_n$ and $\lim b_n$ both exist. In fact, we have

$$\underline{\lim} x_n = \sup_n \inf_{k \geq n} x_k \quad \leq \quad \overline{\lim} x_n = \inf_n \sup_{k \geq n} x_k.$$

For part (ii), we first assume that $l := \lim x_n$ exists. Thus, for any $\varepsilon > 0$, there is a positive integer N such that $l - \varepsilon < x_n < l + \varepsilon$ for all $n \geq N$. Then by the definition of a_n and b_n, we have

$$l - \epsilon \leq a_n \leq b_n \leq l + \varepsilon$$

for all $n \geq N$. Thus, we have $|b_n - a_n| \leq 2\varepsilon$ for all $n \geq N$. By taking $n \to \infty$, this gives $|\overline{\lim} x_n - \underline{\lim} x_n| \leq 2\varepsilon$ for all $\varepsilon > 0$, and hence, we have $\overline{\lim} x_n = \underline{\lim} x_n$.
Now for showing the converse ($\Leftarrow$), we assume that we have $l := \overline{\lim} x_n = \underline{\lim} x_n$. Then for any ε, there is a positive integer N so that $l - \varepsilon < a_n \leq b_n < l + \varepsilon$ for all $n \geq N$. Since we always have $a_n \leq x_k \leq b_n$ for all $k \geq n$. Therefore, we have $l - \varepsilon < x_k < l + \varepsilon$ for all $k \geq N$ and hence, $\lim x_k = l$.
For proving part (iii), we are going to construct a subsequence (x_{n_k}) of (x_n) so that $\lim x_{n_k} = \overline{\lim} x_n$. Let $L := \overline{\lim} x_n$. It is noted that for any $\varepsilon > 0$, there is a positive integer N so that $L - \varepsilon < b_n := \sup_{k \geq n} x_k < L + \varepsilon$ for all $n \geq N$. This implies that

$x_k < L + \varepsilon$ for all $k \geq N$.
If we fix $n \geq N$, since $L - \varepsilon < b_n$ for all $n \geq N$, we can choose $\eta > 0$ such that $L - \varepsilon < b_n - \eta$. Using the characterization of surpemum, we have $L - \varepsilon < b_n - \eta < x_m$ for some $m \geq n$. Therefore, we have shown that

$$\forall \varepsilon > 0, \exists N, \forall n \geq N, \exists m \geq n \quad \text{so that} \quad L - \varepsilon < x_m < L + \varepsilon. \tag{3.1}$$

Now for considering $\varepsilon = 1$ in 3.1, there is N_1 so that $L - 1 < x_{n_1} < L + 1$ for some $n_1 \geq N_1$. Next, for considering $\varepsilon = 1/2$ in 3.1, there is N_2 so that for any $n \geq N_2$, we have $L - 1/2 < x_m < L + 1/2$ for some $m \geq n$. Thus, if we choose $n > N_2$ and $n > n_1$, then there is $n_2 \geq n$ so that $n_2 > n_1$ and $L - 1/2 < x_{n_2} < L + 1/2$.
Similarly, if we take $\varepsilon = 1/3$, there is a positive integer N_3 so that for any $n \geq N_3$ we have $L - 1/3 < x_m < L + 1/3$ for some $m \geq n$. Therefore, if we take $n > N_3$ and $n > n_2$, then there is $n_3 \geq n$ such that $L - 1/3 < x_{n_3} < L + 1/3$ and $n_3 > n_2$.
To repeat the same steps, we get a strictly increasing sequence of positive integers (n_k) so that $L - 1/k < x_{n_k} < L + 1/k$ for all k. Thus, (x_{n_k}) is a convergent subsequence with the limit L. The proof is complete. □

Proposition 3.7 *Let (x_n) and (y_n) be bounded sequences. Then we have*

(i) $\overline{\lim}(-a_n) = -\underline{\lim} a_n$.

(ii) $\overline{\lim}(a x_n) = a(\overline{\lim} x_n)$ *for* $a \geq 0$.

(iii) $\underline{\lim} x_n + \underline{\lim} y_n \leq \underline{\lim}(x_n + y_n) \leq \overline{\lim}(x_n + y_n) \leq \overline{\lim} x_n + \overline{\lim} y_n$

Proof: Parts (i) and (ii) are clear. We want to show part (iii) and claim that

$$\overline{\lim}(x_n + y_n) \leq \overline{\lim} x_n + \overline{\lim} y_n.$$

Let $b := \overline{\lim} x_n$ and $c := \overline{\lim} y_n$. Let $\varepsilon > 0$. Then there is a positive integer N such that $b_n < b + \varepsilon$ and $c_n < c + \varepsilon$ for all $n \geq N$. This implies that $x_k + y_k \leq b_n + c_n < b + c + 2\varepsilon$ for all $k \geq n \geq N$. Therefore, we have $\sup_{k \geq n}(x_k + y_k) < b + c + 2\varepsilon$ for all $n \leq N$ and thus, $\overline{\lim}(x_n + y_n) = \lim_n \sup_{k \geq n}(x_k + y_k) < b + c + 2\varepsilon$ for all $\varepsilon > 0$. This gives $\overline{\lim}(x_n + y_n) = \lim_n \sup_{k \geq n}(x_k + y_k) < b + c$ as desired.
By considering the sequences $(-x_n)$ and $(-y_n)$ in above, we see that $\underline{\lim} x_n + \underline{\lim} y_n \leq \underline{\lim}(x_n + y_n)$. the proof is complete. □

Remark 3.8 It is noted that in general we don't have the equality $\overline{\lim}(x_n + y_n) = \overline{\lim} x_n + \overline{\lim} y_n$. For example, if we let $x_n = (-1)^{n+1}$ and $y_n = (-1)^n$, then $\overline{\lim}(x_n + y_n) < \overline{\lim} x_n + \overline{\lim} y_n$.

4 Compact Sets

Motivated by the Bolzano-Weierstrass Theorem, the following notation plays a very important role in Mathematics.

Definition 4.1 *A subset A of $\mathbb{R}$ is said to be compact if for any sequence (x_n) in A, there is a convergent subsequence (x_{n_k}) of (x_n) such that $\lim x_{n_k} \in A$.*

Example 4.2 *Clearly, $\mathbb{R}$ and $(0, 1)$ are not compact.*

Proposition 4.3 *Every closed and bounded interval is compact.*

Proof: Recall a closed and bounded interval that it is a set $[a, b] := \{x : a \leq x \leq b\}$ for some $-\infty < a < b < \infty$.
Let (x_n) be a sequence in $[a, b]$. Then (x_n) is a bounded sequence. The Bolzano-Weierstrass Theorem gives a convergent subsequence (x_{n_k}). It is noted since $a \leq x_{n_k} \leq b$ for all $k = 1, 2..$, we have $a \leq \lim x_{n_k} \leq b$. Thus, $\lim x_{n_k} \in [a, b]$ as desired. □

Remark 4.4 However, a compact set need not be a closed and bounded interval. For example, $[0, 1] \cup \{2\}$ is a compact set but it is not an interval.

In the remainder of this section, we give a characterization of a compact set.

Definition 4.5 *Let A be a subset of $\mathbb{R}$. A point x is called a limit point (or cluster point) of A if for any $\varepsilon > 0$, there is an element $a \in A$ such that $0 < |x - a| < \varepsilon$, i.e., there is an element $a \in A$ with $x \neq a$ such that $|x - a| < \varepsilon$. We write $D(A)$ for the set of all limit points of A.*
Furthermore, A is said to be closed if $D(A) \subseteq A$.

Example 4.6

(i) If $A = (0, 1] \cup \{2\}$, then $D(A) = [0, 1]$. Hence, A is not closed since $0 \in D(A) \setminus A$.

(ii) If $A = \mathbb{Z}$, then $D(A) = \emptyset$ and thus, $\mathbb{Z}$ is a closed set.

Proposition 4.7 *Let A be a subset of $\mathbb{R}$. Then the following statements are equivalent.*

(i) A is closed.

(ii) If (x_n) is a sequence in A and is convergent, then $\lim x_n \in A$.

Proof: For $(i) \Rightarrow (ii)$, assume that A is closed but the condition (ii) does not hold. Then there is a convergent sequence (x_n) in A but the limit $l := \lim x_n \notin A$. Since A is closed, $D(A) \subseteq A$. Thus, l is not a limit point of A. This implies that there is $\delta > 0$ so that $((l - \delta, l + \delta) \setminus \{l\}) \cap A = \emptyset$. Since $\lim x_n = l$, there is a positive integer N such that $|x_N - l| < \delta$. Note that we have $l \neq x_N$ because $l \notin A$. Hence, $x_N \in ((l - \delta, l + \delta) \setminus \{l\}) \cap A$ which leads to a contradiction. Therefore, (ii) holds.
For $(ii) \Rightarrow (i)$, let $z \in D(A)$. Then for any $\varepsilon > 0$, there is an element $x \in A$ such that $0 < |x - z| < \varepsilon$. Therefore, for each positive integer n, there is an element $x_n \in A$ such that $0 < |x_n - z| < 1/n$ and thus, $z := \lim x_n$. The assumption (i) implies that $z \in A$. Therefore, $D(A) \subseteq A$. The proof is complete. □

Definition 4.8 *For a subset A of $\mathbb{R}$, put*

$$\overline{A} = A \cup D(A).$$

The set $\overline{A}$ is called the closure of A.

Example 4.9 *We have the following examples.*

1. $\overline{(0,1]} = [0,1]$.
2. $\overline{\mathbb{Q}} = \mathbb{R}$.
3. $\overline{\mathbb{Z}} = \mathbb{Z}$.

Proposition 4.10 *Let A be a subset of $\mathbb{R}$. Then we have the following assertions.*

1. *$\overline{A}$ is closed.*
2. *A is closed if and only if $\overline{A} = A$.*
3. *$z \in \overline{A}$ if and only if for any $\delta > 0$, there is an element $a \in A$ so that $|z - a| < \delta$ if and only if there is a convergent sequence (x_n) in A so that $z = \lim x_n$.*
4. *$\overline{A}$ is the smallest closed set containing A, i.e., if B is a closed set containing A, then $\overline{A} \subseteq B$.*

Proof: For showing part (1), we need to show that $D(\overline{A}) \subseteq \overline{A}$. Suppose not, assume that there is an element $z \in D(\overline{A})$ but $z \notin \overline{A}$. Since $z \notin \overline{A}$, there is $\delta > 0$ such that $(z-\delta, z+\delta) \cap A = \emptyset$. On the other hand, there is an element $b \in (z-\delta, z+\delta) \cap \overline{A}$ because $z \in D(\overline{A})$. Now choose $r > 0$ such that $(b-r, b+r) \subseteq (z-\delta, z+\delta)$. Using the definition of limit points again, we can find some element $a \in A$ such that $a \in (b-r, b+r)$ and thus, $a \in (z-\delta, z+\delta) \cap A$. It leads to a contradiction because $(z-\delta, z+\delta) \cap A = \emptyset$ by the choice of δ.
Parts (2)-(4) can be shown by the definition of limit points directly. Try to do it by yourself.
□

Recall that a subset A of $\mathbb{R}$ is said to be dense in $\mathbb{R}$ if for any open interval I, we have $I \cap A \neq \emptyset$.

Proposition 4.11 *Let A be a subset of $\mathbb{R}$. Then A is dense in $\mathbb{R}$ if and only if $\overline{A} = \mathbb{R}$.*

Proof: For showing ($\Rightarrow$): assume that A is a dense set. Let $z \in \mathbb{R}$. Then for any $\delta > 0$, we have $(z-\delta, z+\delta) \cap A \neq \emptyset$ by the definition of a dense set. Hence, there is $a \in A$ such that $|z-a| < \delta$. Thus, $z \in \overline{A}$ by Proposition 4.10(3) above.
Conversely, assume that $\overline{A} = \mathbb{R}$. Let I be an open interval. We want to show $I \cap A$ is non-empty. Fix an element $z \in I$. Since I is an open interval, we can choose $\delta > 0$ such that $(z-\delta, z+\delta) \subseteq I$. Since $\overline{A} = \mathbb{R}$, by using Proposition 4.10(3) again, there is an element $a \in A$ such that $|z-a| < \delta$. Therefore, $a \in (z-\delta, z+\delta) \cap A$ and hence, $I \cap A \neq \emptyset$. The proof is finished. □

Theorem 4.12 *Let A be a subset of $\mathbb{R}$. Then A is compact if and only if A is a closed and bounded subset.*

Proof: For showing the necessary part, we assume that A is compact.
We first claim that A is bounded. Suppose that A is unbounded. If we fix an element $x_1 \in A$, then there is $x_2 \in A$ such that $|x_1 - x_2| > 1$. Using the unboundedness of A, we can find an element x_3 in A such that $|x_3 - x_k| > 1$ for $k = 1, 2$. To repeat the same step, we can find a sequence (x_n) in A such that $|x_n - x_m| > 1$ for $n \neq m$. Thus A has no convergent subsequence. Thus A must be bounded
Finally, we show that A is closed. Let (x_n) be a sequence in A and it is convergent. It needs to show that $\lim_n x_n \in A$. Note that since A is compact, (x_n) has a convergent subsequence (x_{n_k}) such that $\lim_k x_{n_k} \in A$. Then by Proposition 3.3, we see that $\lim_n x_n = \lim_k x_{n_k} \in A$. The proof is finished.
Conversely, we suppose that A is closed and bounded. Let (x_n) be a sequence in A and thus (x_n) is a bounded sequence in $\mathbb{R}$. Then by the Bolzano-Weierstrass Theorem, (x_n) has a convergent subsequence (x_{n_k}). Since A is closed, $\lim_k x_{n_k} \in A$. Therefore, A is compact. $\square$

Example 4.13 *Let $A = \{1/n : n = 1, 2, ...\} \cup \{0\}$. Then A is a compact set.*
A is clearly bounded. Then by Theorem 4.12, it suffices to show that the set A is closed. Clearly, $0 \in D(A)$. We are going to show $D(A) = \{0\}$. In fact, if $z \neq 0$, clearly we can find some $r > 0$ such that the intersection $(z - r, z + r) \cap A$ contains at most one point. Therefore, if $z \neq 0$, then $z \notin D(A)$. Thus, $D(A) = \{0\}$. Hence, the set A is closed as desired.

In the rest of this section, we are going to use another description of a compact set.

For convenience, we call a collection of open intervals $\{J_\alpha : \alpha \in \Lambda\}$ an *open intervals cover* of a given subset A of $\mathbb{R}$, where Λ is an arbitrary non-empty index set, if each J_α is an open interval (not necessary bounded) and

$$A \subseteq \bigcup_{\alpha \in \Lambda} J_\alpha.$$

Theorem 4.14 Heine-Borel Theorem: Any closed and bounded interval $[a, b]$ satisfies the following condition which is called *the Heine-Borel Property.*

(HB) Given any open intervals cover $\{J_\alpha\}_{\alpha \in \Lambda}$ of $[a, b]$, there are finitely many $J_{\alpha_1}, .., J_{\alpha_N}$ such that $[a, b] \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$

Proof: Suppose that $[a, b]$ does not satisfy Heine-Borel Property. Then there is an open intervals cover $\{J_\alpha\}_{\alpha \in \Lambda}$ of $[a, b]$ but it it has no finite sub-cover. Let $I_1 := [a_1, b_1] = [a, b]$ and m_1 the mid-point of $[a_1, b_1]$. Then by the assumption, $[a_1, m_1]$ or $[m_1, b_1]$ cannot be covered by finitely many J_α's. We may assume that $[a_1, m_1]$ cannot be covered by finitely many J_α's. Put $I_2 := [a_2, b_2] = [a_1, m_1]$. To repeat the same steps, we can obtain a sequence of closed and bounded intervals $I_n = [a_n, b_n]$ with the following properties:

(a) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots\cdots;$

(b) $\lim_n (b_n - a_n) = 0$;

(c) each I_n cannot be covered by finitely many J_α's.

Then by the Nested Intervals Theorem, there is an element $\xi \in \bigcap_n I_n$ such that $\lim_n a_n = \lim_n b_n = \xi$. In particular, we have $a = a_1 \leq \xi \leq b_1 = b$. Hence, there is $\alpha_0 \in \Lambda$ such that $\xi \in J_{\alpha_0}$. Since J_{α_0} is open, there is $\varepsilon > 0$ such that $(\xi - \varepsilon, \xi + \varepsilon) \subseteq J_{\alpha_0}$. On the other hand, there is $N \in \mathbb{N}$ such that a_N and b_N in $(\xi - \varepsilon, \xi + \varepsilon)$ because $\lim_n a_n = \lim_n b_n = \xi$. Thus we have $I_N = [a_N, b_N] \subseteq (\xi - \varepsilon, \xi + \varepsilon) \subseteq J_{\alpha_0}$. It contradicts to the Property (c) above. The proof is complete.
□

Remark 4.15 The assumption of the closeness and boundedness of an interval in Heine-Borel Theorem is essential.
For example, notice that $\{J_n := (1/n, 1) : n = 1, 2...\}$ is an open interval covers of $(0, 1)$ but you cannot find finitely many J_n's to cover the open interval $(0, 1)$.

Lemma 4.16 *A subset A is a closed subset of $\mathbb{R}$ if and only if for each element $x \notin A$, there is $r > 0$ such that $(x - r, x + r) \cap A = \emptyset$.*

The following is a very important feature of a compact set.

Theorem 4.17 *Let A be a subset of $\mathbb{R}$. Then the following statements are equivalent.*

(i) For any open intervals cover $\{J_\alpha\}_{\alpha\in\Lambda}$ of A, we can find finitely many $J_{\alpha_1}, .., J_{\alpha_N}$ such that $A \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$.

(ii) A is compact.

(iii) A is closed and bounded.

Proof: The result will be shown by the following path

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i).$$

For $(i) \Rightarrow (ii)$, assume that the condition (i) holds but A is not compact. Then there is a sequence (x_n) in A such that (x_n) has no subsequent which has the limit in A. Put $X = \{x_n : n = 1, 2, ...\}$. Then X is infinite. Note that for each element $a \in A$, there is $\delta_a > 0$ such that $J_a := (a - \delta_a, a + \delta_a) \cap X$ is finite. Indeed, if there is an element $a \in A$ such that $(a - \delta, a + \delta) \cap X$ is infinite for all $\delta > 0$, then (x_n) has a convergent subsequence with the limit a. On the other hand, we have $A \subseteq \bigcup_{a\in A} J_a$. Then by the compactness of A, we can find finitely many $a_1, ..., a_N$ such that $A \subseteq J_{a_1} \cup \cdots \cup J_{a_N}$. Hence, we have $X \subseteq J_{a_1} \cup \cdots \cup J_{a_N}$. Then by the choice of J_a's, X must be finite. This leads to a contradiction. Therefore, A must be compact.
The implication $(ii) \Rightarrow (iii)$ follows immediately from Theorem 4.12.
Finally we want to show $(iii) \Rightarrow (i)$. Suppose that A is closed and bounded. Then we can find a closed and bounded interval $[a, b]$ such that $A \subseteq [a, b]$. Now let $\{J_\alpha\}_{\alpha\in\Lambda}$ be an open intervals cover of A. Note that for each element $x \in [a, b] \setminus A$, there is $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \cap A = \emptyset$ since A is closed by using Lemma 4.16. If we put $I_x = (x - \delta_x, x + \delta_x)$ for $x \in [a, b] \setminus A$, then we have

$$[a, b] \subseteq \bigcup_{\alpha\in\Lambda} J_\alpha \cup \bigcup_{x\in[a,b]\setminus A} I_x.$$

Using the Heine-Borel Theorem 4.14, we can find finitely many J_α's and I_x's, say $J_{\alpha_1}, ..., J_{\alpha_N}$ and $I_{x_1}, ..., I_{x_K}$, such that $A \subseteq [a, b] \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N} \cup I_{x_1} \cup \cdots \cup I_{x_K}$. Note that $I_x \cap A = \emptyset$ for each $x \in [a, b] \setminus A$ by the choice of I_x. Therefore, we have $A \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$ and hence A is compact.
The proof is complete. □

Remark 4.18 In fact, the condition in Theorem 4.17(i) is the usual definition of a *compact set* for a general topological space. More precisely, if a set A satisfies the Definition 4.1, then A is said to be *sequentially compact.* Theorem 4.17 tells us that the notation of the compactness and the sequentially compactness are the same as in the case of a subset of $\mathbb{R}$. However, these two notations are different for a general topological space.

5 Cauchy sequences

The following notation is the landmark in the development of the $20th$ century mathematics.

Definition 5.1 *A sequence (x_n) is called a Cauchy sequence if it satisfies the following condition:*

for any $\varepsilon > 0$, there is a positive integer N so that $|x_m - x_n| < \varepsilon$ whenever $m, n \geq N$.

Remark 5.2 According to the definition of a Cauchy sequence, a sequence (x_n) is not a Cauchy sequence if there is $\varepsilon > 0$ so that for any positive integer N, we can find some $m, n \geq N$ such that $|x_m - x_n| \geq \varepsilon$.

Theorem 5.3 Cauchy Criterion: *A sequence (x_n) is convergent if and only of it is a Cauchy sequence.*

Proof: The necessary part is clear. In fact, if (x_n) is a convergent sequence with the limit L, then for any $\varepsilon > 0$, there is a positive integer N such that $|x_- L| < \varepsilon$ for all $n \geq N$. Therefore, we have

$$|x_m - x_n| \leq |x_m - L| + |L - x_n| < 2\varepsilon \quad \text{as } m, n \geq N.$$

Conversely, we assume that (x_n) is a Cauchy sequence.
We first Claim that (x_n) is a bounded sequence. In fact, since (x_n) is a Cauchy, we can find a positive integer N_1 such that $|x_m - x_{N_1}| < 1$ for all $m \geq N_1$ and thus, $|x_m| < 1 + |x_{N_1}|$ for all $m \geq N_1$. Therefore, we have $|x_m| \leq \max(|x_1|, ..., |x_{N_1 - 1}|, |x_{N_1}| + 1)$ for all positive integers m.
The Bolzano-Weierstrass Theorem tells us that (x_n) has a convergent subsequence (x_{n_k}). Let $L := \lim_k x_{n_k}$. If we show that L is the limit of (x_n), then the proof is finished.
Let $\varepsilon > 0$. Then there is a positive integer N such that $|x_m - x_n| < \varepsilon$ as $m, n \geq N$. On the other hand, since $L = \lim_k x_{n_k}$, we can choose K large enough such that $|L - x_{n_K}| < \varepsilon$ and $n_K > N$. This implies that for any $n \geq N$, we have

$$|x_n - L| < |x_n - x_{n_K}| + |x_{n_K} - L| < 2\varepsilon.$$

The proof is complete. □

Example 5.4 *Let $s_n = \sum_{k=1}^{n} 1/k$. Then (s_n) is not a Cauchy sequence and thus, (s_n) is divergent.*
In fact, it is noted that for $n \leq m$, we have

$$|s_m - s_n| = \frac{1}{n+1} + \cdots + \frac{1}{m} \geq \frac{m-n}{m}.$$

Hence, we always have $|s_{2n} - s_n| \geq \frac{1}{2}$ for all n. Thus, if we take $\varepsilon = 1/2$, then for any positive integer N by taking $n = N$ and $m = 2N$, we have $|s_{2N} - s_N| > 1/2 = \varepsilon$. Hence, (s_n) is not a Cauchy sequence.

Remark 5.5 A sequence (x_n) properly converges to $+\infty$ (resp. $-\infty$) if for any $M > 0$, there is a positive integer N so that $x_n > M$ (resp. $x_n < -M$) for all $n \geq M$. In this case, we write $\lim x_n = \infty$ (resp. $\lim x_n = -\infty$). **Warning**!!! In this case, the sequence (x_n) is still divergent since ∞ is **NOT** a real number, hence, ∞ is not the limit of (x_n).
Note that the sequence (s_n) in Example 5.4 properly converges to $+\infty$. From this we see that the sequence $(\sum_{k=1}^{n} \frac{1}{n^\alpha})_{n=1}^{\infty}$ also diverges properly to $+\infty$ if $\alpha \leq 1$.
However, a divergent sequence may not converge properly to ∞, for example, if we take $x_n = 0$ as n is odd; otherwise, $x_n = n$.

Example 5.6 *Let $t_n = \sum_{k=1}^{n} \frac{1}{k^2}$. Then the sequence (t_n) is convergent. Using the Cauchy Theorem, we need to show that (t_n) is a Cauchy sequence.*
It is noted that for $n \leq m$, we have

$$|t_m - t_n| = \sum_{k=n+1}^{m} \frac{1}{k^2} \leq \sum_{k=n+1}^{m} \frac{1}{(k-1)k} = \sum_{k=n+1}^{m} (\frac{1}{k-1} - \frac{1}{k}) = \frac{1}{n} - \frac{1}{m} < \frac{1}{n}.$$

Thus, if we are given $\varepsilon > 0$, then we choose a positive integer N so that $\frac{1}{n} < \varepsilon$ for all $n \geq N$. Therefore, $|t_m - t_n| < \varepsilon$ whenever $m \geq n \geq N$. The proof is complete.

Remark 5.7 We have the following implications in $\mathbb{R}$.

Axiom of Completeness $\Rightarrow$ Bounded Monotone Convergent Theorem (Theorem 2.13) $\Rightarrow$ Nested Intervals Theroem $\Rightarrow$ Bolzano-Weierstrass Theorem $\Rightarrow$ Cauchy Theorem.

Everything is due to the Axiom of Completeness.

6 Appendix: Bolzano-Weierstrass Theorem and Cauchy Criterion in $\mathbb{R}^m$

Throughout this section, for each element $x \in \mathbb{R}^m$, we write $x := (x(1), ..., x(m))$ and put

$$\|x\| := \sqrt{\sum_{k=1}^{m} |x(k)|^2}.$$

We call $\|x\|$ the norm of x. Clearly, we have

$$\max_{1\le k\le m}|x(k)| \le \|x\| \le \sqrt{m}\max_{1\le k\le m}|x(k)| \tag{6.1}$$

for all $x \in \mathbb{R}^m$.
For each element x and y in $\mathbb{R}^m$, the distance between x and y is defined by $\|x-y\|$. In this case, one can define naturally the notation of a convergent sequence in $\mathbb{R}^m$ as in the $\mathbb{R}$ case. More precisely, we have the following definition.

Definition 6.1 *A sequence (x_n) in $\mathbb{R}^m$ is said to be convergent if there is an element $u \in \mathbb{R}^m$ such that $\lim_n \|x_n - u\| = 0$. Clearly such element is unique if it exists. In this case we call u the limit of (x_n) and write $u := \lim x_n$.*

By using Eq 6.1, we have the following immediately.

Lemma 6.2 *Using the notation as above, a sequence (x_n) converges to u in $\mathbb{R}^m$ if and only if the sequence $(x_n(k))$ converges to $u(k)$ in $\mathbb{R}$ for all $k = 1, ..., m$.*

Naturally, one can also have the following notation as in the $\mathbb{R}$ case.

Definition 6.3 *(i) A point z in $\mathbb{R}^m$ is said to be a limit point of a subset A of $\mathbb{R}^m$ if for every $r > 0$, there is a point $a \in A$ such that $0 < \|z - a\| < r$. Also, A is said to be a closed set if A contains all its limit points.*

(ii) A Cauchy sequence (x_n) in $\mathbb{R}^m$ means that if for every $\varepsilon > 0$, there is a positive integer N such that $\|x_m - x_n\| < \varepsilon$ whenever $m, n \ge N$.

Theorem 6.4 Cauchy Criterion: *A sequence in $\mathbb{R}^m$ is convergent if and only if it is a Cauchy sequence.*

Proof: The Eq 6.1 tells us that if a sequence (x_n) in $\mathbb{R}^m$ is a Cauchy sequence, then so is for each coordinate sequence $(x_n(k))$ in $\mathbb{R}$ for $k = 1, ...m$. Hence the result is obtained immediately by using the Cauchy criterion for the real case. □

In the rest of this section, we are going to show the Bolzano-Weierstrass Theorem in the higher dimensional case. Recall that a subsequence of (x_n) in $\mathbb{R}^m$ is a sequence in $\mathbb{R}^m$ given by a strictly increasing function $\phi : \mathbb{Z}_+ \to \mathbb{Z}_+$, i.e., it is $(x_{\phi(j)})_{j=1}^\infty$.

Theorem 6.5 Bolzano-Weierstrass Theorem: *Every bounded sequence in $\mathbb{R}^m$ has a convergent subsequence.*

Proof: Let (x_n) be a bounded sequence in $\mathbb{R}^m$. Thanks to the Eq6.1 again, each coordinate sequence $(x_n(k))$ is also a bounded sequence of real numbers for $k = 1, ..., m$. As $k = 1$, by the Bolzano-Weierstrass for the real sequence case, there is a convergent subsequence $(x_{\phi_1(j)}(1))_{j=1}^\infty$ of $(x_n(1))$, where $\phi_1 : \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$ is a strictly increasing function. As $k = 2$, we consider the subsequence $(x_{\phi_1(j)})$ of (x_n). Using the Bolzano-Weierstrass for the real sequence case again, there is a convergent subsequence $(x_{\phi_2(j)}(2))_{j=1}^\infty$ of $(x_{\phi_1(j)}(2))$, where $\phi_2 : \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$ is a strictly increasing function. Next we consider the subsequence $(x_{\phi_2\circ\phi_1(j)})$ of $(x_{\phi_1(j)})$, and so is the subsequence of (x_n), for the case of $k = 3$. To repeat the same step, we get a subsequence $(x_{\phi_m\circ\cdots\circ\phi_1(j)})$ of (x_n) so that each coordinate sequence

$(x_{\phi_m \circ \cdots \circ \phi_1(j)}(k))$ is convergent for all $k = 1, ..., m$. Then by the Eq 6.1 again, the subsequence $(x_{\phi_m \circ \cdots \circ \phi_1(j)})$ is convergent in $\mathbb{R}^m$ as desired.

$\square$

7 Limits of functions

Throughout this section let f be a real-valued function defined on a subset A of $\mathbb{R}$.
A point x_0 is called a *limit point* of A if for any $r > 0$, there is some element $a \in A$ such that $0 < |x_0 - a| < r$. We write $D(A)$ for the set of all limit points of A. Note that a limit point of A may not sit in A.

Definition 7.1 *Let $c \in D(A)$. A number L is said to be a limit of f at c (note that $f(c)$ may not be defined!!) if for any ε, there is $\delta = \delta(\varepsilon) > 0$ (depends the choice of ε) such that*

$$|f(x) - L| < \varepsilon \quad \text{whenever } x \in A \text{ and } 0 < |x - c| < \delta \ .$$

(Note: we only consider those points in A which are very close to c but do not equal to c!!!)

Remark 7.2 A number L is not a limit of f at c means if there is $\varepsilon > 0$ so that for any δ, we can find some $x' \in A$ with $|x' - c| < \delta$ but $|f(x') - L| \geq \varepsilon$.

Proposition 7.3 *Using the notation as above if f has a limit at c, then its limit is unique. Consequently, if we write $\lim\limits_{x \to c} f(x)$ for the limit of f at c, then this notation is well defined.*

Proof: Let L' be a another limit of f at c. Let $\varepsilon > 0$. Then by the definition above, there are some positive numbers δ and δ' so that $|f(x) - L| < \varepsilon$ for any $x \in A$ with $0 < |x - c| < \delta$. Similarly, we have $|f(x) - L'| < \varepsilon$ for any $x \in A$ with $0 < |x - c| < \delta'$. Since $c \in D(A)$, we can find some $a \in A$ such that $0 < |c - a| < \delta$", where δ" $= \min(\delta, \delta')$. This gives

$$|L - L'| \leq |L - f(a)| + |f(a) - L'| < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $L = L'$ as desired. $\square$

Example 7.4 *Let $A = (0, \infty)$. Define $f(x) := x^2 \sin \frac{1}{x}$.*

(i) Show that $\lim_{x \to 0} f(x) = 0$.
In fact, it is noted that $|x^2| \leq |x|$ for all $x \in (0, 1)$. Let $\varepsilon > 0$. Thus, if we take $0 < \delta = \min(\varepsilon, 1)$, then we have

$$|f(x) - 0| \leq |x^2| \leq |x| < \varepsilon$$

whenever $x > 0$ with $|x - 0| < \delta$.

(ii) Using the ε-δ notation, show that $\lim_{x \to 0} f(x) \neq 1$.
Note that if we take $\varepsilon = 1/2$, then for any $\delta > 0$, we choose a positive integer N such that $0 < |\frac{1}{N\pi} - 0| < \delta$, and we have

$$|f(\frac{1}{N\pi}) - 1| = 1 > \varepsilon.$$

Therefore, 1 is not the limit of f at 0.

Proposition 7.5 *Using the notation as above, let c be a limit point of A. Then the following are equivalent.*

(i) $\lim_{x\to c} f(x)$ *exists.*

(ii) If (x_n) *is a convergent sequence in* $A\setminus\{c\}$ *with* $\lim x_n = c$*, then the sequence* $(f(x_n))$ *is convergent.*

In this case, $\lim_{x\to c} f(x) = \lim f(x_n)$ *whenever a convergent sequence* (x_n) *in* $A\setminus\{c\}$ *with* $\lim x_n = c$*.*

Proof: For showing $(i) \Rightarrow (ii)$,we assume that $L = \lim_{x\to c} f(x)$ exists. Let (x_n) be a convergent sequence in $A \setminus \{c\}$ with the limit c. Let $\varepsilon > 0$. Then by the definition of the limit of a function, we can find $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in A$ with $0 < |x - c| < \delta$. On the other hand, since $\lim x_n = c$ and $x_n \neq c$ for all n, there is a positive integer N such that $0 < |x_n - c| < \delta$ for all $n \geq N$ and thus, $|f(x_n) - L| < \varepsilon$ for all $n \geq N$. Thus, the condition (ii) holds.
Suppose that the condition (ii) holds. We first claim that the sequence $(f(x_n))$ converges to the same limit whenever (x_n) is a convergent sequence in $A\setminus\{c\}$ such that $\lim x_n = c$. In fact, suppose that there are two sequences (x_n) and (y_n) in $A\setminus\{c\}$ with $\lim x_n = \lim y_n = c$ such that $u := \lim f(x_n)$ and $v := \lim f(y_n)$ are not the same. Put $w_n := x_k$ if $n = 2k$ and $w_n := y_k$ if $n = 2k + 1$ is odd. Then $\lim w_n = c$. The assumption (ii) implies that $\lim f(w_n)$ exists. However, $(f(x_n))$ and $(f(y_n))$ both are the subsequences of $(f(w_n))$ and they converge to the different limits. It leads to a contradiction.
Now we fix a sequence (x_n) in $A \setminus \{c\}$ such that $\lim x_n = c$. Then by the assumption $L := \lim f(x_n)$ exists. Suppose that L is not the limit of $f(x)$ at c. Thus, there is $\varepsilon > 0$ such that for any $\delta > 0$, we can find some $x' \in A$ with $0 < |x' - c| < \delta$ but $|f(x') - L| \geq \varepsilon$. From this, we see that for each positive integer n, there is $x'_n \in A$ with $0 < |x'_n - c| < 1/n$ but $|f(x'_n) - L| \geq \varepsilon$. Thus, the sequence (x'_n) sits in $A\setminus\{c\}$ and converges to c but L is not the limit of the sequence $(f(x'_n))$. This contradicts to the above observation. Therefore, the part (i) holds. □

Proposition 7.5, together with Proposition 2.8, we have the following assertion immediately.

Proposition 7.6 *Let* f *and* g *be the functions defined on* A*. Let* c *be a limit point of* A*. Assume that* $L := \lim_{x\to c} f(x)$ *and* $R := \lim_{x\to c} g(x)$ *both exist. Then we have the following statements.*

1. $\lim\limits_{x\to c}(f + g)(x)$ *exists and* $\lim\limits_{x\to c}(f + g)(x) = L + R$

2. $\lim\limits_{x\to c}(f \cdot g)(x)$ *exists and* $\lim\limits_{x\to c}(f \cdot g)(x) = L \cdot R$.

3. *if we further assume that* $g(x) \neq 0$ *for all* $x \in A$ *and* $R \neq 0$*, then* $\lim\limits_{x\to c}(f/g)(x)$ *exists and* $\lim\limits_{x\to c}(f/g)(x) = L/R$.

The following result is regarded as the Cauchy criterion in the case of functions.

Proposition 7.7 *Using the notation as before,* $\lim_{x\to c} f(x)$ *exists if and only if for any* $\varepsilon > 0$*, there is* $\delta > 0$ *such that* $|f(x') - f(x'')| < \varepsilon$ *whenever* $x', x'' \in A$ *with* $0 < |x' - c| < \delta$ *and* $0 < |x'' - c| < \delta$*.*

Proof: For showing ($\Rightarrow$) we assume that $L := \lim_{x\to c} f(x)$ exists. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that $|f(x) - L| < \varepsilon$ as $x \in A$ with $0 < |x - c| < \delta$. Thus, if $x', x'' \in A$ with $0 < |x' - c| < \delta$ and $0 < |x'' - c| < \delta$, we see that

$$|f(x') - f(x'')| \leq |f(x') - L| + |L - f(x'')| < 2\varepsilon.$$

Hence, the necessary condition holds.
Note that since c is a limit point of A, we can find a sequence Let (x_n) in $A \setminus \{c\}$ such that $\lim x_n = c$. Then the necessary condition implies that $(f(x_n))$ is a Cauchy sequence. In fact, for any $\varepsilon > 0$, the necessary condition above gives $\delta > 0$ so that $|f(x') - f(x'')| < \varepsilon$ whenever $x', x'' \in A$ with $0 < |x' - c| < \delta$ and $0 < |x'' - c| < \delta$. Since $\lim x_n = c$, there is a positive integer N such that $|x_n - c| < \delta$ for all $n \geq N$ and hence, we have $|f(x_n) - f(x_m)| < \varepsilon$ for all $m, n \geq N$. Thus, $(f(x_n))$ is a Cauchy sequence. Hence, $\lim f(x_n)$ exists. The proof is complete by using Proposition 7.5.
□

Definition 7.8 *Using the notation as before, let f be a function defined on A and let c be a limit point of A.*

1. *We say that f diverges to $+\infty$ (resp. $-\infty$) as x tends to c if for any $M > 0$, there is $\delta > 0$ such that $f(x) > M$ (resp. $f(x) < -M$) as $x \in A$ with $0 < |x - c| < \delta$. In this case, write $\lim_{x\to c} f(x) = +\infty$ (resp. $\lim_{x\to c} f(x) = -\infty$).*

2. *We further suppose that A is not bounded above. We say that f has a limit L as x tends to $+\infty$ if for any $\varepsilon > 0$, there is a positive number $R > 0$ such that $|f(x) - L| < \varepsilon$ as $x \in A$ with $x > R$. In this case, a limit must be unique if it exists. Write $\lim_{x\to\infty} f(x) = L$.*
   Similarly, one can define the notion $\lim_{x\to-\infty} f(x) = L$ when A is not bounded below. For simply, when we are talking about notion $\lim_{x\to\infty} f(x)$, A has been assumed to be unbounded above in advance.

3. *Similarly, one can give a suitable definition for the notion: $\lim_{x\to\infty} f(x) = +\infty$.*

Proposition 7.9 *Using the notation as before, let f, g be the functions defined on A.*

(i) If $\lim_{x\to c} f(x) = +\infty$ and $\lim_{x\to c} g(x)$ exists, then $\lim_{x\to c}(f + g)(x) = +\infty$.

(ii) If $\lim_{x\to c} f(x) = +\infty$ and $\lim_{x\to c} g(x) > 0$ exists, then $\lim_{x\to c}(f \cdot g)(x) = +\infty$.

(iii) If $\lim_{x\to\infty} f(x) = +\infty$ and $\lim_{x\to\infty} g(x) > 0$ exists, then $\lim_{x\to\infty}(f \cdot g)(x) = +\infty$.

Proof: For showing part (ii), let $M > 0$. Since $l := \lim_{x\to c} g(x) > 0$, there is $\delta_1 > 0$ so that $g(x) > l - \frac{l}{2} = \frac{l}{2} > 0$ for all $x \in A$ with $0 < |x - c| < \delta_1$. Moreover, $\lim_{x\to c} f(x) = +\infty$, and so we can find $0 < \delta < \delta_1$ such that $f(x) > \frac{2M}{l}$ as $x \in A$ with $0 < |x - c| < \delta$ and hence in this case, we have

$$f(x)g(x) > \frac{2M}{l} \cdot \frac{l}{2} = M.$$

Part (ii) follows.
Using the similar argument, try to finish the proof by yourself. □

Remark 7.10 The assumption of the non-zero limits in Proposition 7.9(ii) and (iii) cannot be removed. For example, by considering $f(x) := 1/x; g(x) := x$ for $x > 0$, note that $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 0} g(x) = 0$ but $f(x)g(x) = 1$ for all $x > 0$.

Example 7.11 *Let $p(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree $n > 0$, where $x \in \mathbb{R}$. If the leading coefficient a_n of p is positive, then $\lim_{x\to+\infty} f(x) = +\infty$. In fact, since $a_n \neq 0$, we see that*

$$p(x) = a_n x^n (1 + \frac{a_{n-1}}{a_n}x^{-1} + \frac{a_{n-2}}{a_n}x^{-2} + \cdots + \frac{a_0}{a_n}x^{-n})$$

for all $x > 0$. In addition, since $a_n > 0$ and $n > 0$, clearly we have $\lim_{x\to+\infty} a_n x^n = +\infty$. The result follows immediately from Proposition 7.9.

Definition 7.12 *A point c is called a right (resp. left) limit point of A if for any $r > 0$, there is some $x \in A$ such that $0 < x - c < r$ (resp. $0 < c - x < r$), i.e., $(c, c + r) \cap A \neq \emptyset$ (resp. $(c - r, c) \cap A \neq \emptyset$). Write $D_r(A)$ (resp. $D_l(A)$) for the set of right (resp. left) limit points of A.*
Clearly, we have $D_r(A) \cup D_l(A) = D(A)$.

Example 7.13 *We have the following examples.*

1. *If $A = (0, 1) \cup \{2\}$, then $D_r(A) = [0, 1)$ and $D_l(A) = (0, 1]$.*
2. *If $A = \{1, 1/2, 1/3, ...\}$, then $D_r(A) = \{0\}$ and $D_l(A) = \emptyset$.*

Definition 7.14 *Using the notation as above, let $c \in D_r(A)$. We say that f has a right (resp. left) limit L of f at c if for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in A$ with $0 < x - c < \delta$ (resp. $0 < c - x < \delta$).*
It is noted that if a right (resp. left) limit exists, then it is unique.
We write $\lim_{x\to c+} f(x)$ and $\lim_{x\to c-} f(x)$ for the right and left limit respectively.

Example 7.15 *Let $A = \mathbb{R} \setminus \{0\}$. Define $f(x) = 1$ if $x > 0$; otherwise, $f(x) = -1$. Then $\lim_{x\to 0+} f(x) = 1$ and $\lim_{x\to 0-} f(x) = -1$. This function is called the sign function.*
We always denote it by $sgn(x)$.

Proposition 7.16 *Let $c \in D_r(A) \cap D_l(A)$. Then $\lim_{x\to c} f(x)$ exists if and only if $\lim_{x\to c+} f(x)$ and $\lim_{x\to c-} f(x)$ both exist and $\lim_{x\to c+} f(x) = \lim_{x\to c-} f(x)$.*
In this case, we have $\lim_{x\to c} f(x) = \lim_{x\to c+} f(x) = \lim_{x\to c-} f(x)$.

Proposition 7.17 *Let $f(x)$ be a function defined on $(0, \infty)$ and $g(x) = f(1/x)$. Then $\lim_{x\to+\infty} f(x)$ exists if and only if $\lim_{x\to 0+} g(x)$ exists.*
In this case, we have $\lim_{x\to+\infty} f(x) = \lim_{x\to 0+} g(x)$.

8 Continuous functions

Throughout this section, let A be a non-empty subset of $\mathbb{R}$ and let f be a function defined on A.

Definition 8.1 *Let $c \in A$. We say that a function f is continuous at c if for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $x \in A$ with $|x - c| < \delta$.*
Furthermore, f is said to be continuous on A if it is continuous at every point in A.

Remark 8.2 Using the notation as above, note that

1. A function f is discontinuous at c if there is $\varepsilon > 0$ so that for any $\delta > 0$, we can find some $x \in A$ satisfying $|x - c| < \delta$ but $|f(x) - f(c)| \geq \varepsilon$.
2. If a point $c \in A$ is not a limit point of A, then a function f is continuous automatically at c. In fact, if $c \in A$ is not a limit point of A, then there is $r > 0$ such that $(c-r, c+r) \cap A = \{c\}$. Therefore, for any $\varepsilon > 0$, we can choose $\delta = r$ in the Definition 8.1 above.

Definition 8.3 *Let A be subset of $\mathbb{R}$. A subset E of A is called an open subset of A if for each $c \in E$, there is $r > 0$ such that $(c - r, c + r) \cap A \subseteq E$.*

Remark 8.4 Warning !!!!: Notice that if E an open subset of A, it does not imply that E is an open subset of $\mathbb{R}$.
For example, if we consider $A = [0, 1]$ and $E = (0, 1]$, then E is an open subset of A but it is not an open subset of $\mathbb{R}$.

The following result is directly obtained from the definition of a continuous function.

Proposition 8.5 *Using the notation as above, then a function f is continuous on A if and only the pre-image $f^{-1}(V) := \{x \in A : f(x) \in V\}$ of any open subset V of $\mathbb{R}$ is an open subset of A.*

Proposition 8.6 *Let $c \in A$. Then we have the following assertions.*

(i) If $c \in A$ is a limit point of A, then f is continuous at c if and only if $\lim_{x \to c} f(x) = f(c)$.

(ii) f is continuous at c if and only if whenever a sequence (x_n) in A with $\lim x_n = c$, we have $\lim f(x_n) = f(c)$.

Proof: Part (i) follows directly from the Definition 8.1.
Part (ii) can be obtained by using a similar argument as in Proposition 7.6. Try to do it by yourself. □

Proposition 8.7 *Let $c \in A$ and let f, g be functions defined on A. If f, g are continuous at c, then we have the following assertions.*

(i) The function $f + g$ is continuous at c.

(ii) The product $f \cdot g$ is continuous at c.

(iii) Moreover, if $g(x) \neq 0$ for all $x \in A$, then f/g is continuous at c.

(iv) Moreover, if the image of f is contained in a subset B of $\mathbb{R}$ and $h : B \to \mathbb{R}$ is continuous at $f(c)$, then the composition $h \circ f$ is continuous at c.

Proof: The above assertions follows immediately from Propositions 2.8 and 8.6. Alternatively, they can be shown directly by the definition.
For showing part (ii), since g is continuous at c, there is $\delta_1 > 0$ such that $|f(x) - f(c)| < 1$ and hence, $|f(x)| < 1 + |f(c)|$ for all $x \in A$ with $|x - c| < \delta_1$. Using the continuity of f and g at c, there exists $0 < \delta < \delta_1$ so that $|f(x) - f(c)| < \varepsilon$ and $|g(x) - g(c)| < \varepsilon$ as $x \in A$ and $|x - c| < \delta$. Therefore, we have

$$|f(x)g(x) - f(c)g(c)| \leq |f(x)g(x) - f(c)g(x)| + |f(c)g(x) - f(c)g(c)| \leq \varepsilon(1 + |g(c)| + |f(c)|)$$

as $x \in A$ and $|x - c| < \delta$. Part (ii) follows.
By using part (ii), we need to show that the function $1/g(x)$ is continuous at c. Note that we may assume $g(c) > 0$ (otherwise by considering $-g(x)$). $g(x)$ is continuous at c, and so there is $\delta_1 > 0$ so that $|g(x) - g(c) < g(c)/2$ and hence, $g(x) > g(c)/2$ for all $x \in A$ and $|x - c| < \delta_1$. Now let $\varepsilon > 0$, there is $0 < \delta < \delta_1$ so that $|g(x) - g(c)| < \delta$ as $x \in A$ and $|x - c| < \delta$. Therefore, we have

$$|\frac{1}{g(x)} - \frac{1}{g(c)}| = \frac{|g(x) - g(c)|}{g(x)g(c)} \leq \frac{2\varepsilon}{g(c)^2}$$

for all $x \in A$ with $|x - c| < \delta$. The proof of (iii) is complete.
The last assertion follows clearly from the definition. □

Before showing the following important result, we first recall that a subset A is said to be compact if for any sequence (x_n) in A has a convergent subsequence (x_{n_k}) such that $\lim_k x_{n_k} \in A$. Moreover, A is compact if and only if it is a closed and bounded set.

Theorem 8.8 *If f is a continuous function defined on a compact set A, then f is a bounded function. Moreover, there are x_1 and x_2 in A such that $f(x_1) = \min\{f(x) : x \in A\}$ and $f(x_2) = \max\{f(x) : x \in A\}$.*

Proof: First, we show that the set $f(A)$ is bounded above. We give two different methods about the claim.
Method I:
Suppose that f is not bounded above. Then for each positive integer n, there is $x_n \in A$ such that $f(x_n) \geq n$. A is compact, and so there is a convergent subsequence (x_{n_k}) with $c := \lim x_{n_k} \in A$. Note that since f is continuous at c, we see that the sequence $(f(x_{n_k}))$ converges to $f(c)$ and thus, $(f(x_{n_k}))$ is a bounded sequence but $f(x_{n_k}) \geq n_k$ for all k. It leads to a contradiction because $n_k \to \infty$ as $k \to \infty$.
Method II:
Notice that since f is continuous on A, then for each element $x \in A$, there is $\delta(x) > 0$ such that $f(u) < f(x) + 1$ as $u \in A$ with $|u - x| < \delta(x)$. Note that if we put $J_x := (x - \delta(x), x + \delta(x))$ for $x \in A$, then we have $A \bigcup_{x \in A} J_x$. Using the compactness of A, there are finitely many $x_1, .., x_N \in A$ such that $A \subseteq J_{x_1} \cup \cdots \cup J_{x_N}$. Then implies that

$f(x) \leq \max(1 + f(x_1), \cdots, 1 + f(x_N))$ for all $x \in A$, so the set $f(A)$ is bounded above.

Next, we want to show that $f(a) = \max\{f(x) : x \in A\}$ for some $a \in A$.
In fact, note that since the set $\{f(x) : x \in A\}$ is bounded above, $L := \sup\{f(x) : x \in A\}$ exists. Thus, there exists a sequence (x_n) in A such that $\lim f(x_n) = L$. Using the compactness of A, there is a convergent subsequence (x_{n_k}) of (x_n) with $a := \lim x_{n_k} \in A$. Thus, we have $f(a) = \lim f(x_{n_k})$ and thus, $f(a) = L$ as desired.
By considering $-f$, we get $f(x_1) = \min\{f(x) : x \in A\}$ for some $x_1 \in A$. The proof is complete. □

Remark 8.9 The assumption of compactness in Theorem 8.8 cannot be removed.
For example if $A = [1, \infty)$ and $f(x) = 1/x$ for $x \in A$, then there is no points attains its minimum on A although f is a bounded function.

Theorem 8.10 *If f is a continuous function defined on a compact set, then the image $f(A) := \{f(x) : x \in A\}$ is compact.*

Proof: Method I:
It suffices to show that $f(A)$ is a closed and bounded set. We have shown that $f(A)$ is bounded by Theorem 8.8. We need to show that $f(A)$ is closed. By applying Proposition 4.7, we need to claim that if (x_n) is a sequence in A so that $(f(x_n))$ is convergent, then the limit $L := \lim f(x_n) \in f(A)$. Indeed, by the compactness of A, (x_n) has a convergent subsequence (x_{n_k}) such that $c := \lim x_{n_k} \in A$. f is continuous at c, and so $\lim f(x_{n_k}) = f(c)$ and thus, $L = f(c) \in f(A)$ as required.
Method II;
Let $\{J_i\}_{i \in I}$ be an open cover of $f(A)$. Since f is continuous on A, Proposition 8.5 implies that for each element $a \in A$, there are $\delta_a > 0$ and $i_a \in I$ such that $f((a - \delta_a, a + \delta_a) \cap A) \subseteq J_{i_a}$.
Note that we have

$$A \subseteq \bigcup_{a \in A} (a - \delta_a, a + \delta_a).$$

Then by the compactness of A, there are finitely many $a_1, ..., a_N$ in A such that

$$A \subseteq \bigcup_{k=1}^{N} (a_k - \delta_{a_k}, a_k + \delta_{a_k}).$$

Therefore, we have

$$f(A) \subseteq \bigcup_{k=1}^{N} f((a_k - \delta_{a_k}, a_k + \delta_{a_k}) \cap A) \subseteq \bigcup_{k=1}^{N} J_{i_{a_k}}.$$

□

Remark 8.11 In general, the image of a closed set under a continuous map is not necessarily closed. For example, $A = [1, \infty)$ and $f(x) = 1/x, x \in A$. Note that A is a closed set but $f(A) = (0, 1]$ is not closed.

Definition 8.12 *Two subsets A and B are said to be homeomorphic if there is a bijection f from A onto B such that f and the inverse f^{-1} both are continuous. In this case, f is called a homeomorphism.*

Proposition 8.13 *Suppose that A and B are homeomorphic. If A is compact, then so is B.*

Proof: It can be shown directly by Theorem 8.10. □

Example 8.14 *By applying Theorem 8.10, it is impossible to find a continuous surjection from $[0,1]$ onto $[0,1)$ because $[0,1]$ is compact but $[0,1)$ is not. Therefore, $[0,1]$ is not homeomorphic to $[0,1)$.*

Proposition 8.15 *Let A and B be non-empty subsets of $\mathbb{R}$. Let $f : A \to B$ be a continuous bijection. If A is compact, then f is a homeomorphism, i.e., the inverse f^{-1} is continuous.*

Proof: Put $y = f(x)$ and $g(y) = f^{-1}(x)$, $x \in A$. Suppose that the function g is discontinuous at some $b \in B$. Then, there is $\varepsilon > 0$ so that for any $\delta > 0$, there is $y \in B$ so that $|y - b| < \delta$ but $|g(y) - g(b)| \geq \varepsilon$. By considering $\delta = 1/n$ for $n = 1, 2, ...$ Therefore, there is a sequence (y_n) in B so that $\lim y_n = b$ and $|g(y_n) - g(b)| \geq \varepsilon$ for all n. Let $x_n = g(y_n) \in A$. Then by the compactness of A, (x_n) has a convergent subsequence (x_{n_k}) such that $a := \lim x_{n_k} \in A$. Note that $b = \lim y_{n_k} = \lim f(x_{n_k}) = f(a)$ because f is continuous and $\lim y_n = b$. Thus, $a = g(b)$. Therefore, we have $\lim g(y_{n_k}) = \lim x_{n_k} = a = g(b)$ which leads to a contradiction because $|g(y_n) - g(b)| \geq \varepsilon$ for all n. □

Remark 8.16 The assumption of compactness of the domain on Proposition 8.15 cannot be removed. For example, by considering $A = [0,1) \cup [1,2]$ and $B = [0,2]$, a function $f : A \to B$ is defined by $f(x) = x$ for $x \in [0,1)$ and $f(x) = x - 1$ for $x \in [1,2]$. Then f is a continuous bijection but its inverse is discontinuous at $y = 1$. Note that A is non-compact in this case.

Theorem 8.17 Intermediate Value Theorem *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Assume that $f(a) < L < f(b)$. Then there is $c \in (a,b)$ such that $f(c) = L$.*

Proof: If we consider the function $x \in [a,b] \mapsto f(x) - L$, then we may assume that $L = 0$, i.e., $f(a) < 0 < f(b)$. We want to show that there is $c \in (a,b)$ so that $f(c) = 0$.
Method 1:
Let $S := \{x \in [a,b] : f(x) > 0$. Note that S is non-empty a bounded below set since $b \in S$ and $x > b$ for all $x \in S$. Thus, $c := \inf S$ exists. We will show that $f(c) = 0$. Note that for each positive integer n, there is $x_n \in S$ satisfying $c \leq x_n < c + 1/n$, and so $\lim x_n = c$. Since $a \leq x_n \leq b$ for all n, we see that $c \in [a,b]$. By the continuity of f and $f(x_n) > 0$ for all n, we have $\lim f(x_n) = f(c)$ and $f(c) \geq 0$. We want to show that it is impossible if $f(c) > 0$. Note that $c > a$ since $f(a) < 0$. Therefore, there is $\delta > 0$ such that $a < c - \delta$ and $|f(x) - f(c)| < f(c)/2$ as $x \in [a,b]$ with $|x - c| < \delta$. Thus, if we fix a point x_1 such that $a < c - \delta < x_1 < c \leq b$, then we have $f(x_1) > f(c)/2 > 0$. This implies that $x_1 \in S$ and $x_1 < c$. It is a contradiction because c is a lower bound for the set S. Therefore, $f(c) = 0$.
Method 2:

Let $[a_1, b_1] = [a, b]$. We want to construct inductively a sequence of closed and bounded intervals $\{[a_k, b_k]\}_{k=1}^{n}$, where $1 \leq n \leq +\infty$, satisfying the following conditions.

1. $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots$.
2. $b_k - a_k = \frac{1}{2}(b_{k-1} - a_{k-1})$, for all $2 \leq k \leq n$.
3. $f(a_k) < 0 < f(b_k)$, for all $1 \leq k \leq n$.

Suppose that the sequence of closed and bounded intervals $([a_k, b_k])$ has been constructed for $1 \leq k \leq n$. We want to construct $[a_{n+1}, b_{n+1}]$ so that it satisfies the conditions $(1) - (3)$ above. Put $m_n := \frac{a_n + b_n}{2}$. If $f(m_n) = 0$, then the result follows. Otherwise, if $f(m_n) > 0$, then we put $[a_{n+1}, b_{n+1}] = [a_n, m_n]$. If $f(m_n) < 0$, then we put $[a_{n+1}, b_{n+1}] = [m_n, b_n]$. Therefore, if $f(m_n) \neq 0$ for all $n = 1, 2...$, then we have an infinite sequence of $([a_k, b_k])$ satisfying the conditions $(1) - (3)$ above. By applying the Nested Intervals Theorem in this case, we have $\bigcap_{k=1}^{\infty} [a_k, b_k] = \{c\}$ for some $c \in [a, b]$. Note that we have $\lim a_k = \lim b_k = c$. f is continuous at c, so we have $f(c) = \lim f(a_k) = \lim f(b_k)$. From this, together with the condition (3) above, we have $f(c) = 0$. The proof is complete. □

Recall that an interval is a non-empty subset of $\mathbb{R}$ which is one of the following forms.

1. (Bounded case): $[a, b]; [a, b); (a, b]$ and (a, b) for $a < b$.
2. (Unbounded case): $[a, +\infty); (a, +\infty); (-\infty, b); (-\infty, b]$ and $\mathbb{R}$.

Proposition 8.18 *Let A be subset of $\mathbb{R}$. Assume that A has at least two points. Then the followings are equivalent.*

1. *A is an interval.*
2. *For each pair of elements $a, b \in A$ with $a < b$, we have $[a, b] \subseteq A$.*

Proof: $(1) \Rightarrow (2)$ is clear. We want to show $(2) \Rightarrow (1)$. Assume that the condition (2) holds.
First, we assume that A is bounded. Then $L := \sup A$ and $l := \inf A$ both exist. Then $x \in [l, L]$ for any $x \in A$, so $A \subseteq [l, L]$. Now, if L and l are in A, then the condition (2) implies that $[l, L] \subseteq A$ and thus, $A = [l, L]$. By using the similar argument for the other cases, i.e., $l \in A$ and $L \notin A$; $l \notin A$ and $L \in A$; $l \notin A$ and $L \notin A$, we see that A is equal to $[l, L)$; $(l, L]$ and (l, L) respectively.
Similarly, the result can be obtained in the unbounded case. □

Theorem 8.19 *Let f be a continuous function defined on A. If A is an interval, then so is its image $f(A)$.*

Proof: By using Proposition 8.18, we need to show that $[c, d] \subseteq f(A)$ whenever $c, d \in f(A)$ with $c < d$. In fact, let $f(a) = c$ and $f(b) = d$ for some $a, b \in A$. We may assume that $a < b$. Note that since A is an interval, we have $[a, b] \subseteq A$. By applying the Intermediate Value Theorem, for any element $L \in [c, d]$, there is an element x_1 between a and b such that $L = f(x_1) \in f(A)$, and hence $[c, d] \subseteq f(A)$. The proof is complete. □

Example 8.20 *By Theorem 8.19, there is no continuous surjections from* $[0,1]$ *onto* $[0,1]\cup[2,3]$. *Hence, the set* $[0,1]$ *is not homeomorphic to* $[0,1]\cup[2,3]$.

9 Uniform continuous functions

Throughout this section, let f be a function defined on a non-empty subset of $\mathbb{R}$.

Definition 9.1 *A function* f *is said to be uniformly continuous on* A *if for any* $\varepsilon > 0$, *there is* $\delta > 0$ *such that* $|f(x) - f(y)| < \varepsilon$ *whenever* $x, y \in A$ *with* $|x - y| < \delta$.

Remark 9.2 A function f is not uniformly continuous on A if there is $\varepsilon > 0$ such that for any $\delta > 0$, there are $x, y \in A$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.

Example 9.3 *(i) Let* $f(x) = x^2$ *for* $x \in [0,\infty)$. *Then* f *is not uniformly continuous on* $[0,\infty)$. *In fact for any positive integer* n, *we have*

$$|f(n+\frac{1}{n}) - f(n)| = (2n + 1/n)(1/n) = 2 + \frac{1}{n^2} \geq 2.$$

Therefore, if we let $\varepsilon = 2$, *then for any* $\delta > 0$, *we choose a positive integer* n *such that* $1/n < \delta$, *so* $n + 1/n$ *and* n *in* $[0,\infty)$ *with* $|n + 1/n - n| < \delta$ *but* $|f(n+\frac{1}{n}) - f(n)| \geq 2$. *Thus,* f *is not uniformly continuous on* $[0,\infty)$.
Note that from this example we see that a continuous function need not be uniformly continuous on its domain.

(ii) Let $f(x) = x^2$ *for* $x \in [0,1]$. *Then* f *is uniformly continuous on* $[0,1]$. *In fact for* $x, y \in [0,1]$ *we have*

$$|f(x) - f(y)| = |x - y||x + y| \leq 2|x - y|.$$

Let $\varepsilon > 0$. *Then we can choose* $0 < \delta < \varepsilon/2$, *so we have* $|f(x) - f(y)| \leq 2|x - y| < \varepsilon$ *whenever* $x, y \in [0,1]$ *with* $|x - y| < \delta$. *Thus,* f *is uniformly continuous on* $[0,1]$.

Theorem 9.4 *Let* f *be a continuous function on* A. *If* A *is compact, then* f *is uniformly continuous on* A.

Proof: Method I:
Suppose that A is compact but f is not uniformly continuous on A. Then there is $\varepsilon > 0$ such that for any $\delta > 0$, there are $x, y \in A$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$. Consider $\delta = 1/n$ for $n = 1, 2, ...$ Then for any positive integer n, there are x_n and y_n such that $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \varepsilon$.
Then by the compactness of A, the sequence (x_n) has a convergent subsequence (x_{n_k}) such that $a := \lim_k x_{n_k}$. By applying the compactness of A, the sequence (y_{n_k}) has a convergent subsequence $(y_{n_{k_i}})$ with $b := \lim_i y_{n_{k_i}} \in A$. Note that we still have $a := \lim_i x_{n_{k_i}}$. Since $|x_{n_{k_i}} - y_{n_{k_i}}| < 1/n_{k_i}$ for all $i = 1, 2, ...$, we have $a = b$. Hence, we have

$$\lim_i f(x_{n_{k_i}}) = f(a) = f(b) = \lim_i f(y_{n_{k_i}}),$$

and so we have

$$0 < \varepsilon \leq |f(x_{n_{k_i}}) - f(y_{n_{k_i}})| \to 0 \quad \text{as } i \to \infty.$$

It leads to a contradiction.

Method II:

Let $\varepsilon > 0$. Let $a \in A$. Since f is continuous at a, there is $\delta_a > 0$ so that $|f(x) - f(a)| < \varepsilon$ as $x \in A$ and $|x - a| < \delta_a$. Put $J_a := (a - \frac{\delta_a}{2}, a + \frac{\delta_a}{2})$. Then we have $A \subseteq \bigcup_{a \in A} J_a$. By using the compactness of A, there are finitely many $a_1, ..., a_N \in A$ such that $A \subseteq \bigcup_{k=1}^{N} J_{a_k}$. Take $0 < \delta < \frac{1}{2}\delta_{a_k}$ for all $k = 1, ..., N$. Let $x, x' \in A$ with $|x - x'| < \delta$. Note that $x \in J_{a_i}$ for some $1 \le i \le N$. Then we have $|x - a_i| < \frac{1}{2}\delta_{a_i} < \delta_{a_i}$ and $|x' - a_i| \le |x' - x| + |x - a_i| < \delta_{a_i}$. Thus, we have

$$|f(x) - f(x')| \le |f(x) - f(a)| + |f(a) - f(x')| < 2\varepsilon.$$

The proof is complete. □

Theorem 9.5 Weierstrass approximation theorem *Let $f(x)$ be a real-valued continuous function defined on $[0,1]$. Then for any $\varepsilon > 0$, there is a polynomial function $p(x)$ such that $|f(x) - p(x)| < \varepsilon$ for all $x \in [0,1]$.*

Proof: Before showing the result, let us recall a class of polynomial functions in the following, called *Bernstein polynomials.* For each positive integer n, set

$$B_n(x) := \sum_{k=0}^{n} a_k \binom{n}{k} x^k (1-x)^{n-k}, \qquad \text{where } a_k \in \mathbb{R}$$

for $x \in [0,1]$. Using the binomial theorem, clearly we have

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1 \tag{9.1}$$

for all $x \in [0,1]$. Taking $x(1-x)\frac{d}{dx}$ in Eq 9.1, we have

$$\sum_{k=0}^{n} \binom{n}{k} [x^k (1-x)^{n-k}](k - nx) = 0.$$

Again taking $\frac{1}{n^2} x(1-x)\frac{d}{dx}$ on both side and applying the product rule of derivatives, we have

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (x - \frac{k}{n})^2 = \frac{x(1-x)}{n}.$$

This gives

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (x - \frac{k}{n})^2 \le \frac{1}{4n} \quad \text{for all } x \in [0,1] \tag{9.2}$$

since $\max\{x(1-x) : x \in [0,1]\} = 1/4$.

Now let

$$p_n(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(\frac{k}{n}), \qquad \text{for } x \in [0,1].$$

The theorem is obtained if we show that for any $\varepsilon > 0$, there is a positive integer N such that $|f(x) - p_n(x)| < \varepsilon$ for all $n \geq N$ and for all $x \in [0, 1]$. Note that using the Eq9.1, we see that

$$|f(x) - p_n(x)| \leq \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} |f(x) - f(\frac{k}{n})| \tag{9.3}$$

for all $x \in [0, 1]$ and for all $n = 1, 2....$ On the other hand, since f is continuous on $[0, 1]$, f is uniformly continuous on $[0, 1]$. Thus, for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(u) - f(v)| < \varepsilon$ whenever $u, v \in [0, 1]$ with $|u - v| < \delta$. Now for any $x \in [0, 1]$ and $n = 1, 2, ...$, we put

$$A := \sum_{k:|x-\frac{k}{n}|<\delta} \binom{n}{k} x^k (1-x)^{n-k} |f(x) - f(\frac{k}{n})|$$

and

$$B := \sum_{k:|x-\frac{k}{n}|\geq\delta} \binom{n}{k} x^k (1-x)^{n-k} |f(x) - f(\frac{k}{n})|.$$

Then the Eq 9.3 can be written as the following

$$|f(x) - p_n(x)| \leq A + B.$$

Notice that by the choice of δ, we have

$$A \leq \varepsilon \sum_{k:|x-\frac{k}{n}|<\delta} \binom{n}{k} x^k (1-x)^{n-k} = \varepsilon.$$

We are now going to estimate the value B. By using Eq 9.2, we have

$$\delta^2 \sum_{k:|x-\frac{k}{n}|\geq\delta} \binom{n}{k} x^k (1-x)^{n-k} \leq \sum_{k:|x-\frac{k}{n}|\geq\delta} \binom{n}{k} x^k (1-x)^{n-k} (x - \frac{k}{n})^2 \leq \frac{1}{4n}.$$

This gives

$$\sum_{k:|x-\frac{k}{n}|\geq\delta} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{1}{4n\delta^2}.$$

Since f is continuous, f is bounded on $[0, 1]$. If we put $M := \sup |f(x)|$, then we can now conclude that

$$B \leq 2M \sum_{k:|x-\frac{k}{n}|\geq\delta} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{M}{2n\delta^2}$$

for all $x \in [0, 1]$ and for all $n = 1, 2, ...$ Therefore, if we choose a positive integer N so that $\frac{M}{2n\delta^2} < \varepsilon$ for all $n \geq N$, then $|f(x) - p_n(x)| \leq A + B < 2\varepsilon$ for all $n \geq N$ and for all $x \in [0, 1]$ as desired. The proof is complete. □

Definition 9.6 *Let A be a non-empty subset of $\mathbb{R}$. A function $f : A \to \mathbb{R}$ is called a Lipschitz function if there is a constant $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in A$. In this case.*
Furthermore, if we can find such $0 < C < 1$, then we call f a contraction.

Clearly we have the following property.

Proposition 9.7 *Every Lipschitz function is uniformly continuous on its domain.*

Example 9.8 *(i) : The sine function $f(x) = \sin x$ is a Lipschitz function on $\mathbb{R}$ since we always have $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.*

(ii) : Define a function f on $[0, 1]$ by $f(x) = x \sin(1/x)$ for $x \in (0, 1]$ and $f(0) = 0$. Then f is continuous on $[0, 1]$ and thus f is uniformly continuous on $[0, 1]$, but note that f is not a Lipschitz function. In fact, for any $C > 0$, if we consider $x_n = \frac{1}{2n\pi+(\pi/2)}$ and $y_n = \frac{1}{2n\pi}$, then $|f(x_n) - f(y_n)| > C|x_n - y_n|$ if and only if

$$\frac{2}{\pi} \cdot \frac{(2n\pi + \frac{\pi}{2})(2n\pi)}{2n\pi + \frac{\pi}{2}} = 4n > C.$$

Therefore, for any $C > 0$, there are $x, y \in [0, 1]$ such that $|f(x) - f(y)| > C|x - y|$ and hence f is not a Lipschitz function on $[0, 1]$.

Proposition 9.9 *Let A be a non-empty closed subset of $\mathbb{R}$. If $f : A \to A$ is a contraction, then there is a unique fixed point of f, i.e., there is a point $a \in A$ such that $f(a) = a$.*

Proof: First we show the existence. f is a contraction on A, so there is $0 < C < 1$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in A$. Fix $x_1 \in A$. Since $f(A) \subseteq A$, we can inductively define a sequence (x_n) in A by $x_{n+1} = f(x_n)$ for $n = 1, 2...$ Note that we have

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq C|x_n - x_{n-1}|$$

for all $n = 2, 3...$ This gives

$$|x_{n+1} - x_n| \leq C^{n-1}|x_2 - x_1|$$

for $n = 2, 3, ...$ Thus, for any $n, p = 1, 2..$, we see that

$$|x_{n+p} - x_n| \leq \sum_{i=n}^{n+p-1} |x_{i+1} - x_i| \leq |x_2 - x_1| \sum_{i=n}^{n+p-1} C^{i-1}.$$

Since $0 < C < 1$, for any $\varepsilon > 0$, there is N such that $\sum_{i=n}^{n+p-1} C^{i-1} < \varepsilon$ for all $n \geq N$ and $p = 1, 2, ...$ Therefore, (x_n) is a Cauchy sequence and thus the limit $a := \lim_n x_n$ exists. A is closed, so we have $a \in A$ and hence f is continuous at a. On the other hand, since $x_{n+1} = f(x_n)$, we have $a = f(a)$ by taking $n \to \infty$.
Finally, we show the uniqueness of the fixed point. In fact, if a and b are the fixed points of f and $a \neq b$, then we have $|a - b| = |f(a) - f(b)| \leq C|a - b| < |a - b|$ because $0 < C < 1$. It leads to a contradiction. The proof is complete. The proof is complete. □

Remark 9.10 Proposition 9.9 does not hold if f is not a contraction. For example, if we consider $f(x) = x - 1$ for $x \in \mathbb{R}$, clearly we have $|f(x) - f(y)| = |x - y|$ and f has no fixed point in $\mathbb{R}$.

Proposition 9.11 *Let f be a continuous function defined on (a, b). The the followings are equivalent.*

(i) There exists a continuous function $F : [a, b] \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in (a, b)$.

(ii) f is uniformly continuous on (a, b).

(iii) The limits $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ both exist.

In this case, this continuous extension F is uniquely determined by f. In fact, $F(a) = \lim_{x\to a+} f(x)$ and $F(b) = \lim_{x\to b-} f(x)$.

Proof: For $(i) \Rightarrow (ii)$, we assume that (i) holds. Then by Theorem 9.4, F is uniformly continuous on $[a, b]$, so $f = F|_{(a,b)}$ is uniformly continuous on (a, b).
For $(ii) \Rightarrow (iii)$, we are going to show that $\lim_{x\to b-} f(x)$ exists.
It suffices to show that the sequence $(f(x_n))$ converges to the same limit whenever any sequence (x_n) in (a, b) that converges to b.
First, we claim that $(f(x_n))$ is a Cauchy sequence for any such sequence (x_n) in (a, b). Let $\varepsilon > 0$. Then by the assumption (ii), there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ as $x, y \in (a, b)$ with $|x - y| < \delta$. Now since $\lim x_n = b$ and thus (x_n) is a Cauchy sequence. Therefore, we can find a positive N such that $|x_m - x_n| < \delta$ when $m, n \geq N$. This gives $|f(x_m) - f(x_n)| < \varepsilon$ as $m, n \geq N$. The claim follows and thus, the limit $\lim_{n\to\infty} f(x_n)$ exists.
Next we want to show that if (x_n) and (y_n) both are the sequences in (a, b) that converge to b, then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$. Let $L = \lim_{n\to\infty} f(x_n)$ and $L' = \lim_{n\to\infty} f(y_n)$. Let $\varepsilon > 0$ and let δ be given by the uniform continuity of f. Since $\lim x_n = \lim y_n$, we can choose a positive integer N large enough so that $|x_N - y_N| < \delta$. In addition, such N satisfies $|f(x_N) - L| < \varepsilon$ and $|f(y_N) - L'| < \varepsilon$ because $L = \lim_{n\to\infty} f(x_n)$ and $L' = \lim_{n\to\infty} f(y_n)$. This implies that

$$|L - L'| \leq |L - f(x_N)| + |f(x_N) - f(y_N)| + |f(y_N) - L'| < 3\varepsilon$$

for all $\varepsilon > 0$. Thus, $L = L'$ and hence, the limit $\lim_{x\to b-} f(x)$ exists.
The proof of the case $\lim_{x\to a+} f(x)$ is similar.
Finally, we show $(iii) \Rightarrow (i)$. Define $F(a) := \lim_{x\to a+} f(x)$; $F(b) := \lim_{x\to b-} f(x)$ and $F(x) := f(x)$ for $x \in (a, b)$. Note that F is continuous on $[a, b]$. In fact, we have $F(a) = \lim_{x\to a+} f(x) = \lim_{x\to a+} F(x)$ and $F(b) = \lim_{x\to b-} f(x) = \lim_{x\to b-} F(x)$. Thus, F is continuous at $x = a$ and b.
The last assertion follows immediately from the continuity of F. The proof is complete. □

Remark 9.12 Indeed, in the proof of Proposition 9.11 $(i) \Rightarrow (ii)$ above, we have shown the following fact. Suppose that f is uniformly continuous function defined on A. If (x_n) is a Cauchy sequence in A, then so is the sequence $(f(x_n))$. We can use this simple observation to see a function "NOT" being uniformly continuous on its domain.
Note the assumption of the uniform continuity of f is essential in here by considering the simple example that $f(x) = \frac{1}{x}$, $x \in A := (0, 1]$ and $x_n = \frac{1}{n}$, $n = 1, 2....$

Definition 9.13 *A function $s : [a, b] \to \mathbb{R}$ is called a step function (resp. piecewise linear) if there exist finitely many points $a = x_0 < x_1 < \cdots < x_n = b$ such that s is a constant on each (x_{k-1}, x_k) (resp. linear on $[x_{k-1}, x_k]$, i.e, $s(x) = m_k x + b_k$) for all $k = 1, .., n$.*

Proposition 9.14 *If f is a continuous function defined on a closed and bounded interval $[a,b]$, then it can be uniformly approximated by step functions (resp. piecewise linear functions), that is, for each $\varepsilon > 0$, there exists a step function s (resp. piecewise linear function) defined on $[a,b]$ such that $|f(x) - s(x)| < \varepsilon$ for all $x \in [a,b]$.*

Proof: By using Theorem 9.4, we first note that f is uniformly continuous on $[a,b]$. Let $\varepsilon > 0$. Then there is $\delta > 0$ so that $|f(x)-f(y)| < \varepsilon$ whenever $x, y \in (a,b)$ with $|x-y| < \delta$. If e choose a partition $a = x_0 < \cdots < x_n = b$ on $[a,b]$ such that $|x_k - x_{k-1}| < \delta$ for $k = 1, ..., n$. Now if we let $s(x) := f(x_{k-1})$ when $x \in [x_{k-1}, x_k)$, then s is the step function as desired. Using the similar argument, the result is obtained for the case of piecewise linear functions. □

10 Monotone Functions

Using the notation given as before, f is a function defined on a subset A of $\mathbb{R}$. f is called a *monotone function* if it is either increasing or decreasing. The following results also hold for decreasing functions by considering $-f$ instead. Recall that c is a *right (resp. left) limit point* of A if for any $r > 0$ we have $(c, c+r) \cap A \neq \emptyset$ (resp. $(c-r, c) \cap A \neq \emptyset$).

Proposition 10.1 *Let f be an increasing function on A. Let $c \in A$. Put*

$$L(c) := \inf\{f(x) : x \in A, x > c\} \quad \textit{if} \quad \{x \in A, x > c\} \neq \emptyset.$$

Similarly, we put

$$l(c) := \sup\{f(x) : x \in A, x < c\} \quad \textit{if} \quad \{x \in A : x < c\} \neq \emptyset.$$

If c is a right (resp. left) limit point of A, then $L(c) = f(c+) := \lim\limits_{x \to c+} f(x)$ (resp. $l(c) = f(c-) := \lim\limits_{x \to c-} f(x)$).

Proof: First, we want to prove that if c is a right limit point of A, then the right limit $f(c+)$ exists. Since c is a right limit point of A, $\{f(x) : x \in A, x > c\} \neq \emptyset$. f is increasing, so $f(c)$ is a lower bound of the set $\{f(x) : x \in A, x > c\}$. The Axiom of Completeness implies that $L(c) := \inf\{f(x) : x \in A, x > c\}$ exists and $f(c) \leq L(c)$. Thus, for any $\varepsilon > 0$, there is $x_1 \in A$ with $x_1 > c$ such that $f(x_1) < L(c) + \varepsilon$. Hence, if we take $0 < \delta < x_1 - c$, then $L(c) - \varepsilon < L(c) \leq f(x) \leq f(x_1) < L(c) + \varepsilon$ whenever $x \in (c, c+\delta)$. Thus, $L(c) = f(c+)$ as desired.
The proof for the case of a left limit point is similar. □

Proposition 10.2 *Using the notation given as in Proposition 10.1, let f be a strictly increasing (not necessary continuous) function defined on an interval I, i.e, $f(x_1) < f(x_2)$ if and only if $x_1 < x_2$ as $x_1, x_2 \in I$. Let $g : f(I) \longrightarrow I$ be the inverse of f. If $d \in f(I)$, then $g(d) = L(d) =: \inf\{g(y) : y \in f(I), y > d\}$ (resp. $g(d) = l(d) := \sup\{g(y) : y \in f(I), y > d\}$) provided $L(d)$ (resp. $l(d)$) exists.*
In addition, if d is a right (resp. left) limit point of $f(I)$, then $g(d) = g(d+)$ (resp. $g(d) = g(d-)$).
Consequently, the inverse function $g : f(I) \to I$ is continuous.

Proof:

Note that g is also strictly increasing on $f(I)$. Let $c := g(d)$, hence $c \in I$ and $f(c) = d$. g is increasing, so $g(d) \leq L(d)$ whenever $L(d)$ exists. We now suppose that $g(d) < L(d)$, thus we can choose a point z such that $c = g(d) < z < L(d)$. Then by the definition of $L(d)$, there is $y_1 \in f(I)$ with $y_1 > d$. Thus, we have $z < L(d) \leq g(y_1)$. If we let $x_1 = g(y_1)$, then $x_1 \in I$ and $c < z < L(d) \leq x_1$. I is an interval, so $z \in (c, x_1) \subseteq I$. Thus, $f(z) > f(c) = d$, so $f(z) \in \{y \in f(I) : y > d\}$. This implies that $z = g(f(z)) \geq L(d)$. It leads to a contradiction because $c < z < L(d)$ by the choice of z. Therefore, $g(d) = L(d)$.
Similarly, we also have a contradiction if $l(d) < g(d)$. Hence $l(d) = g(d)$.
Finally, we want to show that g is continuous at d in the following cases.
If d is an isolated point of $f(I)$, then g is automatically continuous at d.
If d is a right limit point of $f(I)$ but is not a left limit point of $f(I)$, then by Proposition 10.1, we have $g(d) = L(d) = g(d+)$. Therefore, g is continuous at d. Similarly, if d is a left limit point of $f(I)$ but is not a right limit point of $f(I)$, then we have $g(d) = l(d) = g(d-)$, hence g is continuous at d.
Finally, if d is a right and left limit point of $f(I)$. Then, we have $g(d) = g(d+) = g(d-)$ and so g is continuous at d. The proof is complete. □

Proposition 10.3 *Let f be an increasing function defined on A and let D be the set of discontinuous points of f. Then D is a countable set.*

Proof: For each integer n, we put $D_n := \{x \in D : n-1 \leq f(x) \leq n\}$. Then $D = \bigcup_{n\in\mathbb{Z}} D_n$. Therefore, it suffices to show that each D_n is countable.
We now fix D_m. By using Proposition 10.1, we first note that $c \in D_m$ if and only if $f(c) - f(c-) > 0$ or $f(c+) - f(c) > 0$. Put $J(c-) := [f(c-), f(c)]$ and $J(c+) := [f(c), f(c+)]$. Then $J(c+)$ or $J(c-)$ is an interval. Therefore, if we put $\alpha(c)$ is the length of $(J(c-) \cup J(c+))$ for $c \in D_m$, then $\alpha(c) > 0$. On the other hand, if $c_1, c_2 \in D_m$ with $c_1 < c_2$, then $J(c_1+) \cap J(c_2-)$ has at most one point if they exist. Thus, we have

$$0 < \sum_{c \in D_m} \alpha(c) \leq m - (m-1) = 1.$$

Since $\alpha(c) > 0$ for all $c \in D_m$, the set D_m need to be countable. In fact, note that we have

$$D_m = \bigcup_{k \in \mathbb{Z}_+} \{c \in D_m : \alpha(c) \geq 1/k\}.$$

Thus, if D_m is uncountable, then there exists a positive integer k so that $R := \{c \in D_m : \alpha(c) \geq 1/k\}$ is infinite. Therefore, $\sum_{c \in R} \alpha(c)$ is infinite. It leads to a contradiction. □

Let I be an interval. We call a function $f : I \longrightarrow \mathbb{R}$ *locally bounded* at $c \in I$ if there is $r > 0$ such that the function f is bounded on $(c - r, c + r) \cap I$. Notice that if $\lim\limits_{x \to c} f(x)$ exists, then f is locally bounded at c.

Proposition 10.4 *Now if f is a locally bounded at $c \in I$, then the following always exist.*

$$\overline{\lim_{x \to c}} f(x) := \lim_{\delta \to 0+} \Big(\sup_{0 < |x-c| < \delta} f(x) \Big) \quad \text{and} \quad \underline{\lim}_{x \to c} f(x) := \lim_{\delta \to 0+} \Big(\inf_{0 < |x-c| < \delta} f(x) \Big).$$

Moreover, we have $\underline{\lim}_{x \to c} f(x) \leq \overline{\lim}_{x \to c} f(x)$.

Proof: For simply, we assume that I is an open interval.
For $\delta > 0$, put $L(\delta) := \sup_{0<|x-c|<\delta} f(x)$ and $\ell(\delta) := \inf_{0<|x-c|<\delta} f(x)$. Since f is locally bounded at c, $L(\delta)$ and $\ell)\delta)$ are defined for arbitrary small $\delta > 0$. Moreover, the functions $L(\delta)$ is increasing and $\ell(\delta)$ is decreasing. Then by Proposition 10.1, we see that $\overline{\lim}_{x\to c} f(x) = \inf_{\delta>0} (\sup_{0<|x-c|<\delta} f(x))$ and $\underline{\lim}_{x\to c} f(x) = \sup_{\delta>0} (\inf_{0<|x-c|<\delta} f(x))$.
On the other hand, since $\ell(\delta) \le L(\delta)$ for all $\delta > 0$, we have $\underline{\lim}_{x\to c} f(x) \le \overline{\lim}_{x\to c} f(x)$. $\square$

Example 10.5 *Define* $f(x) = \sin\frac{1}{x}$ *for* $x \neq 0$. *Then* $\overline{\lim}_{x\to 0} f(x) = 1$ *and* $\underline{\lim}_{x\to 0} f(x) = -1$. *Hence, the case* $\underline{\lim}_{x\to 0} f(x) < \overline{\lim}_{x\to 0} f(x)$ *may occur.*

Proposition 10.6 *Using the notation as in Proposition 10.4, the following are equivalent.*

(i) $lim_{x\to c} f(x)$ *exists.*

(ii) $\underline{\lim}_{x\to c} f(x) = \overline{\lim}_{x\to c} f(x)$.

In this case, we have $lim_{x\to c} f(x) = \underline{\lim}_{x\to c} f(x) = \overline{\lim}_{x\to c} f(x)$.

Proof: Let $L := lim_{x\to c} f(x)$ if it exists.
For showing $(i) \Rightarrow (ii)$, let $\varepsilon > 0$, then there is $\delta > 0$ such that $L - \varepsilon < f(x) < L + \varepsilon$ whenever $x \in I$ with $0 < |x - c| < \delta$. Using the notation as in Proposition 10.4, then we have $L - \varepsilon \le \ell(\delta_1) \le L(\delta_1) \le L + \varepsilon$ for all $0 < \delta_1 < \delta$. Taking $\delta_1 \to 0+$, we have

$$L - \varepsilon \le \underline{\lim}_{x\to c} f(x) \le \overline{\lim}_{x\to c} f(x) \le L + \varepsilon$$

for all $\varepsilon > 0$. This gives $\underline{\lim}_{x\to c} f(x) = \overline{\lim}_{x\to c} f(x) = L$ as desired.
$(ii) \Rightarrow (i)$ follows immediately from the simple observation that $\ell(\delta) \le f(x) \le L(\delta)$ for all $\delta > 0$ and for all $x \in I$ with $0 < |x - c| < \delta$. $\square$

Definition 10.7 *A function* $f : [a, b] \longrightarrow \mathbb{R}$ *is said to be of bounded variation if it satisfies the following condition.*

$$\|f\|_{BV} := \sup\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| : a = x_0 < \cdots < x_n = b\} < \infty.$$

We call $\|f\|_{BV}$ *the total variation of* f *and write* $BV[a, b]$ *for the set of all functions of bounded variation.*

Proposition 10.8 *We keep all notation as given above. Then we have*

(i) The space $BV[a, b]$ *is a vector space.*

(ii) Every function of bounded variation is bounded.

(iii) Every monotone function is of bounded variation.

(iv) Every Lipschitz function is of bounded variation.

Proof: Parts $(i), (iii)$ and (iv) are clear.
For showing Part (ii), let $f \in BV[a,b]$ and $x \in (a,b)$. Then by considering the partition $a < x < b$, we have $|f(x) - f(a)| + |f(b) - f(x)| \leq \|f\|_{BV}$. This gives $2|f(x)| \leq |f(a)| + |f(b)| + \|f\|_{BL}$ for all $x \in (a,b)$ and hence, f is bounded on $[a,b]$. □

Remark 10.9 The part (iii) in Proposition 10.8 tells us that a function of bounded variation may not be continuous. On the other hand, a bounded function is not necessary to be of bounded variation.

Example 10.10 *Let*

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \in (0, \frac{2}{\pi}]; \\ 0 & \text{if } x = 0. \end{cases}$$

Notice that f is bounded on $[0, \frac{2}{\pi}]$ but $f \notin BV[0, \frac{2}{\pi}]$. In fact, for each positive integer n, if we let $x_k := \frac{1}{\frac{\pi}{2} + (n-k)\pi}$ for $1 \leq k \leq n$ and consider the partition on $[0, \frac{2}{\pi}] : 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = \frac{2}{\pi}$, then we have

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \geq \sum_{k=2}^{n} |f(x_k) - f(x_{k-1})| \geq 2(n-1)$$

for all $n \geq 2$. This implies that f is not of bounded of variation.

Proposition 10.11 *Suppose that $f \in BV[a,b]$. Put*

$$\phi(x) = \begin{cases} \sup\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| : a = x_0 < \cdots < x_n = x\} & \text{if } x \in (a,b]; \\ 0 & \text{if } x = a. \end{cases}$$

Then the functions ϕ and $\phi - f$ are increasing on $[a,b]$. Consequently, a function $f : [a,b] \longrightarrow \mathbb{R}$ is of bounded variation if and only if $f = u - v$ for some increasing functions u and v on $[a,b]$.

Proof: We first notice that since $f \in BV[a,b]$, the function ϕ is well defined. In fact, we have $0 \leq \phi(x) \leq \|f\|_{BV}$ for all $x \in [a,b]$. Let $a < x < y \leq b$. For any partition on $[a,x]$: $a = x_0 < \cdots < x_n = x$, we see that

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| + |f(y) - f(x)| \leq \phi(y).$$

This implies that $\phi(x) \leq \phi(y)$ and

$$\phi(x) \leq \phi(y) - |f(y) - f(x)| \leq \phi(y) - (f(y) - f(x)).$$

Thus, we have

$$\phi(x) - f(x) \leq \phi(y) - f(y).$$

Therefore, the function $\phi - f$ is increasing on $[a,b]$.
For the last assertion, if $f \in BV[a,b]$, then we have $f = \phi - (\phi - f)$ an so f can be written as the difference of increasing functions as desired. Conversely, since $BV[a,b]$ is a vector space and every increasing function is of bounded variation, the difference of increasing functions is a function of bounded variation. The proof is complete. □

References

[1] R.G. Bartle and R. Sherbert, Introduction to real analysis, 4th edition. John Wiley & Sons, Inc. (2011).