MATH 2058 Mathematical Analysis I
2023-24 Term 1
Suggested Solution to Homework 8

5.4-6 Show that if f and g are uniformly continuous on $A \subseteq \mathbb{R}$ and if they are *both* bounded on A, then their product fg is uniformly continuous on A.

Solution. Since f and g are bounded on A, there exists $M > 0$, such that $|f|, |g| \leq M$ on A. Let $\varepsilon > 0$. Since f and g are both uniformly continuous on A, there exists $\delta > 0$ such that if $x, u \in A$ and $|x - u| < \delta$, then

$$|f(x) - f(u)| < \frac{\varepsilon}{2M} \quad \text{and} \quad |g(x) - g(u)| < \frac{\varepsilon}{2M} \quad .$$

Hence, if $x, u \in A$ and $|x - u| < \delta$, then

$$\begin{aligned}
|fg(x) - fg(u)| &= |f(x)g(x) - f(x)g(u) + f(x)g(u) - f(u)g(u)| \\
&\leq |f(x)g(x) - f(x)g(u)| + |f(x)g(u) - f(u)g(u)| \\
&= |f(x)|\,|g(x) - g(u)| + |g(u)|\,|f(x) - f(u)| \\
&\leq M\,|g(x) - g(u)| + M\,|f(x) - f(u)| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Therefore, fg is uniformly continuous on A. □

5.4-7 If $f(x) := x$ and $g(x) := \sin x$, show that both f and g are uniformly continuous on $\mathbb{R}$, but that their product fg is not uniformly continuous on $\mathbb{R}$.

Solution. First, we show f, g are all uniformly continuous. Note that

$$|f(x) - f(u)| = |x - u|$$

and

$$|g(x) - g(u)| = |\sin x - \sin u| = 2|\sin\frac{x-u}{2}\cos\frac{x+u}{2}| \leq 2\left|\frac{x-u}{2}\right| = |x - u|\,.$$

So, given any $\varepsilon > 0$, if we choose $\delta = \varepsilon$, then whenever $x, u \in \mathbb{R}$ with $|x - u| < \delta$, we have

$$|f(x) - f(u)| < \varepsilon, \quad |g(x) - g(u)| < \varepsilon.$$

Hence, f, g are both uniformly continuous on $\mathbb{R}$.

But fg is not uniformly continuous. Consider $x_n := 2\pi n, u_n := 2\pi n + \frac{1}{n}$ for $n \in \mathbb{N}$. Clearly, we have $\lim(x_n - u_n) = \lim \frac{1}{n} = 0$, and

$$|f(x_n)g(x_n) - f(u_n)g(u_n)| = \left|0 - (2\pi n + \frac{1}{n})\sin\frac{1}{n}\right| \geq 2\pi n \sin\frac{1}{n}.$$

By the well-known limit $\lim_{x\to 0}\frac{\sin x}{x} = 1$, we have

$$\lim 2\pi n \sin\frac{1}{n} = 2\pi.$$

So for n large enough, say $n > N$ for some integer N, we will have $2\pi n \sin\frac{1}{n} \geq \pi$, and thus

$$|f(x_n)g(x_n) - f(u_n)g(u_n)| \geq \pi.$$

By Nonuniform Continuity Criteria, we know fg is not uniformly continuous on $\mathbb{R}$.

□

5.4-8 Prove that if f and g are each uniformly continuous on $\mathbb{R}$, then the composite function $f \circ g$ is uniformly continuous on $\mathbb{R}$.

Solution. Let $\varepsilon > 0$. Since f is uniformly continuous on $\mathbb{R}$, there exists $\delta_1 > 0$ such that

$$|f(u) - f(v)| < \varepsilon \qquad \text{whenever } u, v \in \mathbb{R} \text{ with } |u - v| < \delta_1.$$

Since g is uniformly continuous on $\mathbb{R}$, there exists $\delta_2 > 0$ such that

$$|g(x) - g(y)| < \delta_1 \qquad \text{whenever } x, y \in \mathbb{R} \text{ with } |x - y| < \delta_2.$$

Now, if $x, y \in \mathbb{R}$ and $|x - y| < \delta_2$, then

$$|(f \circ g)(x) - (f \circ g)(y)| = |f(g(x)) - f(g(y))| < \varepsilon.$$

Therefore $f \circ g$ is uniformly continuous on $\mathbb{R}$. □