MATH 2058 Mathematical Analysis I
2023-24 Term 1
Suggested Solution to Homework 7

5.2-6 Let f, g be defined on $\mathbb{R}$ and let $c \in \mathbb{R}$. Suppose that $\lim_{x\to c} f = b$ and that g is continuous at b. Show that $\lim_{x\to c} g \circ f = g(b)$. (Compare this result with Theorem 5.2.7 and the preceding exercise.)

Solution. Let $\varepsilon > 0$. Since g is continuous at b, we can find $\delta > 0$ such that for all $|x - b| < \delta$, we have $|g(x) - g(b)| < \varepsilon$. Since $\lim_{x\to c} f = b$, so for $\varepsilon' = \delta$, we can find $\delta' > 0$ such that if $0 < |x - c| < \delta'$, we have $|f(x) - b| < \delta$. Hence, if $0 < |x - c| < \delta'$, we have $|f(x) - b| < \delta$ and so

$$|(g \circ f)(x) - g(b)| = |g(f(x)) - g(b)| < \varepsilon.$$

Therefore $\lim_{x\to c} g \circ f = g(b)$.

In the preceding exercise, g is not continuous at $1 = f(0)$ and $\lim_{x\to 0} g \circ f \neq (g \circ f)(0)$. □

5.2-7 Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is discontinuous at every point of $[0, 1]$ but such that $|f|$ is continuous on $[0, 1]$.

Solution. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ -1 & \text{if } x \text{ is irrational.} \end{cases}$$

Then f is discontinuous everywhere in $[0, 1]$. Given any $c \in [0, 1]$, we can find a sequence (x_n) of rational numbers and a sequence (y_n) irrational numbers such that $\lim(x_n) = \lim(y_n) = c$ but $\lim(f(x_n)) = 1$ while $\lim(f(y_n)) = -1$. Thus f is discontinuous at c by Discontinuity Criterion.

On the other hand, $|f(x)| = 1$ for any $x \in [0, 1]$, and a constant function is clearly continuous. □

5.2-15 Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous at a point c, and let $h(x) := \sup\{f(x), g(x)\}$ for $x \in \mathbb{R}$. Show that $h(x) = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|$ for all $x \in \mathbb{R}$. Use this to show that h is continuous at c.

Solution. Define $l(x) := \inf\{f(x), g(x)\}$. We claim that for any $x \in \mathbb{R}$,

$$h(x) + l(x) = f(x) + g(x), \quad h(x) - l(x) = |f(x) - g(x)|.$$

If $f(x) \leq g(x)$, then $h(x) = g(x)$, $l(x) = f(x)$ and the formulas follow; if $g(x) < f(x)$, then $h(x) = f(x)$, $l(x) = g(x)$, and we have the same formulas similarly. Hence

$$h(x) = \frac{1}{2}[h(x) + l(x) + h(x) - l(x)] = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|$$

Since both f and g are continuous at $x = c$, and $x \mapsto |x|$ is continuous on $\mathbb{R}$, it follows from Proposition 8.7 that h is continuous at $x = c$. □