MATH 2058 Mathematical Analysis I
2023-24 Term 1
Suggested Solution to Homework 6

5.1-2 Establish the Discontinuity Criterion 5.1.4.

Solution. The Discontinuity Criterion follows immediately from the Sequential Criterion for Continuity by taking negation. We will prove the Sequential Criterion for Continuity.

Sequential Criterion for Continuity. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in A$. Then the following are equivalent.*

(i) f is continuous at c.

(ii) for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.

Suppose f is continuous at c and (x_n) is a sequence in A that converges to c. Let $\varepsilon > 0$. By the definition of continuity, we can find $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $x \in A$ with $|x - c| < \delta$. On the other hand, since $\lim(x_n) = c$, there is $N \in \mathbb{N}$ such that $|x_n - c| < \delta$ for all $n \geq N$ and hence, $|f(x_n) - f(c)| < \varepsilon$ for all $n \geq N$. Thus, the condition (ii) holds.

Suppose the condition (ii) holds but f is not continuous. Then there is $\varepsilon_0 > 0$ such that for any $\delta > 0$, we can find $x' \in A$ with $|x' - c| < \delta$ but $|f(x') - f(c)| \geq \varepsilon_0$. From this, we see that for each $n \in \mathbb{N}$, there is $x'_n \in A$ with $|x'_n - c| < 1/n$ but $|f(x'_n) - f(c)| \geq \varepsilon_0$. Thus, the sequence (x'_n) lies in A and converges to c but $(f(x'_n))$ does not converge to $f(c)$. This contradicts the condition (ii). $\square$

5.1-8 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $S := \{x \in \mathbb{R} : f(x) = 0\}$ be the "zero set" of f. If (x_n) is in S and $x = \lim(x_n)$, show that $x \in S$.

Solution. By the Sequential Criterion for Continuity, $\lim f(x_n) = f(x)$. Since (x_n) is in S, we have $f(x_n) = 0$ for all n. Therefore $f(x) = 0$, that is $x \in S$. $\square$

5.1-12 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $f(r) = 0$ for every rational number r. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

Solution. Fix $x \in \mathbb{R}$. By the Density Theorem, there exists a sequence (x_n) so that

$$x_n \in \mathbb{Q} \cap (x, x + \frac{1}{n}), \quad \text{for all } n \in \mathbb{N}.$$

Thus $\lim(x_n) = x$ by the Squeeze Theorem. Since f is continuous at x and $f(x_n) = 0$ for all $n \in \mathbb{N}$, the Sequential Criterion for Continuity implies that

$$f(x) = \lim f(x_n) = 0.$$

Since x is arbitrary, we have $f(x) = 0$ for all $x \in \mathbb{R}$. $\square$