MATH 2058 Mathematical Analysis I
2023-24 Term 1
Suggested Solution to Homework 5

4.1-9 Use either the ε-δ definition of limit or the Sequential Criterion for limits, to establish the following limits.

(c) $\lim_{x\to 0} \frac{x^2}{|x|} = 0.$

Solution. Method I: ε-δ definition of limit.

For any $\varepsilon > 0$, we choose $\delta = \varepsilon$. If $0 < |x - 0| < \delta$, then we have

$$\left| \frac{x^2}{|x|} - 0 \right| = \frac{|x|^2}{|x|} = |x| < \delta = \varepsilon.$$

Hence

$$\lim_{x\to 0} \frac{x^2}{|x|} = 0.$$

Method II: Sequential Criterion for limits.

If (x_n) is a sequence of real numbers that converges to 0 such that $x_n \neq 0$ for all $n \in \mathbb{N}$, we have

$$\lim \left(\frac{x_n^2}{|x_n|} \right) = \lim(|x_n|) = 0.$$

By the Sequential Criterion for limits, $\lim_{x\to 0} \frac{x^2}{|x|} = 0.$ □

4.1-12 Show that the following limits do *not* exist.

(c) $\lim_{x\to 0}(x + \operatorname{sgn}(x))$.

Solution. Denote $f(x) := x + \operatorname{sgn}(x)$, $x \in \mathbb{R}$. Let (a_n), (b_n) be two sequences defined by $a_n = \frac{1}{n}$, $b_n = -\frac{1}{n}$ for $n \in \mathbb{N}$. Then $a_n, b_n \neq 0$ for all $n \in \mathbb{N}$, and $\lim(a_n) = \lim(b_n) = 0$. However, $\lim(f(a_n)) = \lim(\frac{1}{n} + 1) = 1$ while $\lim(f(b_n)) = \lim(-\frac{1}{n} - 1) = -1$. By the Sequential Criterion (Proposition 7.5), $\lim_{x\to 0} f(x)$ does not exist. □

4.1-15 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) := x$ if x is rational, and $f(x) = 0$ if x is irrational.

(a) Show that f has a limit at $x = 0$.

(b) Use a sequential argument to show that if $c \neq 0$, then f does not have a limit at c.

Solution. (a) Given $\varepsilon > 0$, set $\delta = \delta(\varepsilon) = \varepsilon$. If $0 < |x - 0| < \delta$, then either $|f(x) - 0| = |x| < \varepsilon$ if x is rational or $|f(x) - 0| = 0 < \varepsilon$ if x is irrational. Thus f has limit $L = 0$ at $x = 0$.

(b) In order to show the divergence, we show that for any $c \neq 0$ there exist two sequences (a_n) and (b_n) converging to c while $\lim f(a_n) \neq \lim f(b_n)$.

By the density of rational numbers (Proposition 1.12), for each $n \in \mathbb{N}$, there exists $a_n \in \mathbb{Q}$ such that $c < a_n < c + \frac{1}{n}$. Then $\lim a_n = c$ by the Squeeze Theorem. Note that $f(a_n) = a_n$ and so $\lim f(a_n) = c$.

On the other hand, by the density of irrational numbers (Theorem 1.15), for each $n \in \mathbb{N}$, there exists $b_n \in \mathbb{R} \backslash \mathbb{Q}$ such that $c < b_n < c + \frac{1}{n}$. Similarly we have $\lim b_n = c$. Note that $f(b_n) = 0$ and so $\lim f(b_n) = 0$.

Since $\lim f(a_n) \neq \lim f(b_n)$, f does not have a limit at c.

□