MATH 2058 Mathematical Analysis I
2023-24 Term 1
Suggested Solution to Homework 4

3.5-2 Show directly from the definition that the following are Cauchy sequences.

(a) $\left(\dfrac{n+1}{n}\right)$, (b) $\left(1+\dfrac{1}{2!}+\cdots+\dfrac{1}{n!}\right)$.

Solution. (a) Let $\varepsilon > 0$. Take $N \in \mathbb{N}$ such that $1/N < \varepsilon$. Then, for $n > m \geq N$, we have

$$\left|\frac{n+1}{n} - \frac{m+1}{m}\right| = \frac{n-m}{nm} \leq \frac{n}{nm} = \frac{1}{m} \leq \frac{1}{N} < \varepsilon.$$

Therefore it is a Cauchy sequence.

(b) Let $\varepsilon > 0$. Take $N \in \mathbb{N}$ such that $1/N < \varepsilon$. Then, for $n > m \geq N$, we have

$$\begin{aligned}\left|\left(1+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)-\left(1+\frac{1}{2!}+\cdots+\frac{1}{m!}\right)\right| &= \frac{1}{(m+1)!}+\cdots+\frac{1}{n!}\\ &\leq \frac{1}{2^m}+\cdots+\frac{1}{2^{n-1}}\\ &= \frac{1}{2^m}\left(1+\cdots+\frac{1}{2^{n-m-1}}\right)\\ &\leq \frac{1}{2^{m-1}} \leq \frac{1}{m} \leq \frac{1}{N} < \varepsilon.\end{aligned}$$

Therefore it is a Cauchy sequence.

□

3.5-5 If $x_n := \sqrt{n}$, show that (x_n) satisfies $\lim |x_{n+1} - x_n| = 0$, but it is not a Cauchy sequence.

Solution. Note that

$$x_{n+1} - x_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \quad \text{for all } n \in \mathbb{N}.$$

Hence $\lim |x_{n+1} - x_n| = 0$.

However, (x_n) is not a Cauchy sequence because $|x_{4n} - x_n| = \sqrt{n} \geq 1$ for all $n \in \mathbb{N}$. □

3.5-9 If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is a Cauchy sequence.

Solution. Note that, for $n, k \in \mathbb{N}$, we have

$$\begin{aligned}|x_{n+k} - x_n| &\leq |x_{n+k} - x_{n+k-1}| + \cdots + |x_{n+1} - x_n|\\ &< r^{n+k-1} + \cdots + r^n\\ &= r^n\left(r^{k-1}+\cdots+1\right)\\ &= r^n \cdot \frac{1-r^k}{1-r}\\ &\leq \frac{r^n}{1-r}.\end{aligned}$$

Let $\varepsilon > 0$. Since $0 < r < 1$, we have $\lim(r^n) = 0$, and thus there exists $N \in \mathbb{N}$ such that $r^n < (1-r)\varepsilon$ whenever $n \geq N$. Now, for $n \geq N$ and $k \in \mathbb{N}$, we have

$$|x_{n+k} - x_n| \leq \frac{r^n}{1-r} < \varepsilon.$$

Therefore (x_n) is a Cauchy sequence. $\square$